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MODELLING NETWORK DEGRADATION USING GAME THEORY — OPTIMAL STRATEGIES AT NASH EQUILIBRIUM

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ABSTRACT

Analysing system-performance at the *worst-case* can give very valuable inputs for infrastructure planning. In this study, we formulate and solve network link degradation problem at worst-operable-case (WOC) — a case with maximum system travel time (STT) with the demand being met. For this, we employ *game theory* and envisage the situation as a game between two *n*-member teams on an *n*-linked network. One team tries to increase the STT by each player deciding how much to degrade the link capacities. The other team tries to decrease the STT by deciding the flow on the links. The *Nash equilibrium* defines a set of competitive optimal strategies that both teams have to adopt for them to have maximum payoffs without cooperation. We present the scenario at such Nash equilibrium as the WOC of a managed urban road network; and the link flow vector at the Nash equilibrium as the best routing scheme at the WOC. We formulate a *minimax* optimisation equation and prove that it can solve to optimal strategies at Nash equilibrium even in the absence of probabilistic mixing of strategies. The model is illustrated using two simple networks with an origin and a destination. A two-space genetic algorithm, available in the literature, is used to solve the minimax optimisation problem. It is found that the model solves to WOC, which is logical with respect to not only the link attributes, but also the network topology.

Keywords: network degradation, worst-operable-case, link capacity degradation vector, non-cooperative game theory, Nash equilibrium

INTRODUCTION

There are many instances in general engineering design which require worst-case analysis (Antonsson and Otto, 1995). In the specific area of network design, the notions of resilience, robustness, sustainability, reliability etc. are growing more into popularity. It is in these contexts that analysing performance at worst-case has better potential in the case of transportation networks. The performance at a hypothetical worst-case can be used to compare future alternatives and to measure the present effectiveness of a network. For instance, the ratio between a worst-case and a best-case performance has been identified as a measure of effectiveness of a system (Koutsoupias and Papadimitriou, 1999). Worst-case performance has also been used to identify the reliability of a network (Bell, 2000) (Satayapiwat, Suksomboon and Aswakul, 2008). In the area of computer architecture, high worst-case throughput is one among the desirable objectives in interconnection network design (Seo et al., 2005).

Previous works on worst-case service conditions in networks have adopted standard results on stochastic games (Maitra and Parthasarathy, 1970) (Kumar and Shiau, 1981) (Altman and Shimkin, 1993) (Bell, 2000). Koutsoupias and Papadimitriou (1999) introduced the worst-case Nash equilibrium for a simple special network, which was a set of *m* parallel links connecting an origin to a destination. Worst-case of networks with parallel links were further examined by Czumaj and Vöcking (2002), where they investigated the bounds of worst-case equilibria. Most of the works after Koutsoupias and Papadimitriou (1999) focus on worst-cases resultant of selfish and spontaneous actions of the network users, but not of uncertainities in the performance parameters. Even though Bell (2000) had presented a methodology to model worst expectable cost on a transportation network at deterministic link capacity degradations, the area of worst-case due to *stochastic* link capacity degradation is under-researched. We present here, a methodology to arrive at a hypothetical worst-case due to stochastic link capacity degradations on a network at which the demand remains met.

Worst-Operable-Case (WOC) of an Urban Road Network

Urban road networks are subject to stochastic link capacity degradations due to traffic incidents, flooding, storming, snowing, road space reallocations, road space infiltrations etc. and combinations thereof. They increase the travel time on the network, and there will be a system travel time (STT) which will be the highest possible one, with demand being met. By *demand*, we mean the number of vehicles that wish to travel. We call this upper bound of travel time as the WOC of an urban road network.

A relatively easy maximisation problem should be able to find such a *link capacity degradation vector* that would result in a WOC for the network. However, the WOC of an urban network cannot solely be defined on the degradations in link capacities. The response of the network users to the degradations, in terms of routing, also has to be accounted. We employ the system-optimal (SO) routing scheme, in which users are assumed to selflessly cooperate to optimize the performance of the network.
Methodology and Results

PROBLEM DEFINITION, METHODOLOGY AND RESULTS

Consider a network $\mathscr{G}(\mathscr{N},\mathscr{A})$ where $\mathscr{N}$ is the set of nodes and $\mathscr{A}$ is the set of arcs. $\mathscr{R}$ denotes the set of origins and $\mathscr{S}$ denotes the set of destinations. The problem is to find a vector of post-disruption capacity retention ratio Y, that forms the WOC of the network. WOC is found out considering both the link capacity degradation and the system-optimal routing simultaneously. The link degradations have to be constrained on an upper bound such that the demand on the network remains catered. The routing is done system-optimally. The problem is hence parallel— to find a routing pattern that is the system-optimal response to a WOC link capacity degradation vector, which in turn depends on the routing pattern.

The link capacity degradation vector has to be determined to maximise STT and the routing to minimise it. The problem is approached using *non-cooperative game theory*, as there are two entities who try to compete each other to better their respective objectives. Moreover, this is a *zero-sum game*, as both of the entities are trying to minimise and maximise the STT. In the game, one entity tries to increase the STT by deciding how much to degrade the link capacities. The other entity tries to decrease the STT by deciding the flow on the links. Hence, the link flow vector forms the strategy of the entity playing *for* the network, and the link capacity degradation vector forms the strategy of the entity playing against. The solution can be found by optimising both entities' strategies. At the optimal strategy, both of the entities would reach an equilibrium that they will not be able to minimise (or maximise) the STT any further. The optimal strategy of the entity playing against the network will be the link capacity degradation vector that causes WOC scenario.

We present here a formulation for the above-mentioned problem. The strategies played by both players are optimized using a minimax formulation, which finds the minimum possible value of an objective function with respect to one set of decision variables that coexists with the maximum possible value of the same function with respect to a different set of decision variables. Following are the other notations associated with the formulation (note that link specific variables exist only if the corresponding link exists):

q^{rs}	:	trip rate demand between the origin r and destination s,($r\in\mathscr{R}\, s\in\mathscr{S}$)
ij	:	directed link incident to node i and j; $ij\in\mathscr{A}$
x_{ij}	:	flow on link ij
T_{ij}	:	free-flow travel time on link ij
K_{ij}	:	achievable capacity on link ij
Y	:	vector of post-disruption capacity retention ratio
y_{ij}	:	divisive factor on link capacity— elements of Y
α, β	:	parameters of the cost function
Q_i	:	demand rate from/to the node i if $i\in\mathscr{R}/\mathscr{S}$, zero otherwise

The minimax optimization formulation is presented below.

$$z = \min_{x_{ij}} \max_{y_{ij}} \sum_{ij} x_{ij}.c_{ij}\left(x_{ij}, y_{ij}\right) \tag{1}$$

Subject to:

$$\sum_{j} x_{ij} - \sum_{j} x_{ji} + Q_i = 0 \quad \forall\, i \tag{2}$$

$$0 \le x_{ij} \le \left(K_{ij} / y_{ij}\right) \;\forall ij \tag{3}$$

$$1 \le y_{ij} \le m_{ij} \quad \forall ij \tag{4}$$

where m_{ij} is a finitely large value such that $\left(100\left(m_{ij}-1\right)/m_{ij}\right)\%$ is the maximum allowable capacity degradation on any link ij. It can be seen that:

$$Q_i = \begin{cases} \sum_s q^{rs} & i = r \in R \\ \sum_r q^{rs} & i = s \in S \\ 0 & i \notin R, i \notin S \end{cases} \tag{5}$$

We use $c_{ij}\left(x_{ij}, y_{ij}\right) = T_{ij} + \beta_{ij}\left(\frac{x_{ij}}{\left(K_{ij}/y_{ij}\right)}\right)^{\alpha_{ij}}$ in the numerical examples.

The formulation is illustrated on three test networks and SO-WOC was determined for these networks. Test network 1 is shown in Figure 1. The demand rate was assumed to be 6 vehicles per unit time from node 1 to node 4.A free flow travel time of five time units and a capacity of 12 vehicles per unit time were assumed on all links. The multiplicative coefficient and the power coefficient of the cost function were respectively assumed as 0.15 and one. Though only one OD pair have been considered in this example, applicability of the formulation on cases with multiple OD pairs have been shown using test network 2 and test network 3.

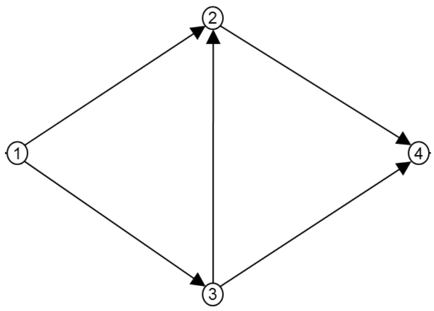

Figure 1 - Test network 1

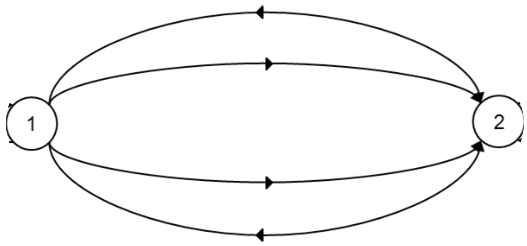

Figure 2 - Test network 2

Test network 2 is shown in Figure 2. There are two OD pairs; with both the nodes 1 and 2 being an origin and a destination each. There are four links— two paths in either direction, with only one link in each of them. Demand rate of 10 vehicles per unit time is assumed on both the OD pairs. The free flow travel time is assumed as one time-unit, and the capacity 10 vehicles per unit time on all links. The multiplicative coefficient and the power coefficient of the cost function are respectively assumed as 10 and one.

The minimax optimisation problem was solved using a *two-space genetic algorithm* developed by Hermann (1999). The population was divided into two subpopulations based on their interest (i.e. to minimise or to maximise) on the objective function. Initial subpopulations of x_{ij} and y_{ij} were randomly generated within the feasible region. For each individual of a population, the objective function value was calculated with respect to each individual of the other population. The objective function in Equation (1) was used to define the *fitness function* for both the populations such that the fitness function penalizes higher values of the function for x_{ij} and rewards higher values of the function for y_{ij}. This fitness function decides the parents in the present generation from whom the elite mutated and reproduced children were born into the next generation. This process was continued till a specified number of generations. The link capacity degradation vector at WOC and other results obtained after performing the computations are presented in Table 4.

Table 4– Results from test networks 1 and 2

Test network	Link	Start-End nodes *(i-j)*	Flow (x_{ij})	Divisive factor (y_{ij})	Percentage degradation	Total link travel time	SO-WOC STT
1	1	*1-2*	3.000000	4.000000	75.00000	15.45000	66.95
	2	*1-3*	3.000000	4.000000	75.00000	15.45000	
	3	*3-2*	1.000000	12.00008	91.66672	5.150001	
	4	*2-4*	4.000000	3.000000	66.66667	20.60000	
	5	*3-4*	2.000000	6.000000	83.33333	10.30000	
2	1_u[1]	*1-2*	7.209000	1.387155	27.91000	79.29900	220.0
	1_l	*1-2*	2.791000	3.582945	72.09000	30.70100	
	2_u	*2-1*	9.581000	1.043732	4.190000	105.3910	
	2_l	*2-1*	0.419000	23.86654	95.81003	4.609034	

The percentage degradation against each link in the above table would effect in the SO-WOC of the network. However, it must be noted that even though the minimax equilibrium value (SO-WOC STT) is unique as demonstrated in proposition 3, the strategies (X^*, Y^*) need not be. This property is well-known in minimax optimization, and many similar results can be found in the literature. (Salmon 1968) (Didinsky, Basar & Bernhard 1993) (Luss 1999) (Heikkinen 1999) (Bell 2000). The absence of uniqueness in the strategy set need not be a concern as the planner will only be interested in the final value of the SO-WOC STT.

CONCLUSIONS

The paper introduces taxonomy of SO-WOC. A methodology is outlined to model the SO-WOC on an urban transportation network with congestion. We believe that the performance at SO-WOC of urban transportation networks can be handy in comparing alternatives and analysing effectiveness of the system. For instance, an option which has a better SO-WOC performance can be considered robust and reliable than another. Therefore, the parameter SO-WOC will be appealing for a planner when there is exclusively stipulated budgetary provision for robustness and reliability. We propose the SO-WOC STT as the maximum possible value of STT caused due to link capacities degradation in a demand-meeting network under SO routing. The methodology envisages a game between two players, playing for and against the network, and finds the Nash equilibrium of the game. A minimax optimization problem is formulated which solves to a unique SO-WOC STT. The formulation has been exemplified on test networks where the minimax optimization problem was solved using a two-stage genetic algorithm.

Link capacity degradations are constrained on an expected maximum level of degradation in this work. However, practitioners may improve the realistic flavour by defining distribution functions on link capacity degradations, if such data is available. The imminent issue researchable is concerning the uniqueness of the optimal strategy sets at the SO-WOC since comparison of various alternatives with respect to SO-WOC tends to be computationally hard for larger networks in the absence of uniqueness on strategies.

[1] The subscripts *u* and *l* differentiates the upper and lower link between a same pair of start-end nodes

REFERENCES

Altman, E. and N. Shimkin (1993). Worst-case and Nash routing policies in parallel queues with uncertain service allocations, IMA Preprint Series, vol. 1120.

Antonsson, E.K. and K.N. Otto (1995). Imprecision in Engineering Design, ASME Journal of Mechanical Design, vol. 117(B), 25-32.

Bell, M.G.H. (2000). A game theory approach to measuring the performance reliability of transport networks, Transportation Research Part B, 533-545.

Czumaj, A. and B. Vöcking (2002). Tight bounds for worst-case equilibria, Proceedings of the thirteenth annual ACM-SIAM symposium on Discrete algorithms, 413-420.

Dantzig, G.B., Harvey, R.P., Lansdowne, Z.F., Robinson, D.w. and Maier, S.F. (1979) 'Formulating and solving the network design problem by decomposition', Transportation Research Part B: Methodological, vol. 13, no. 1, pp. 5-17.

Didinsky, G., Basar, T. and Bernhard, P. (1993) 'Structural properties of minimax policies for a class of differential games arising in nonlinear H∞- control and filtering', Proceedings of the 32nd IEEE Conference on Decision and Control, San Antonio, 184-189.

Herrmann, J.W. (1999). A genetic algorithm for minimax optimisation problems, Proceedings of the 1999 Congress on Evolutionary Computation, Washington, DC, 1099-1103.

Koutsoupias, E. and C. Papadimitriou, (1999). Worst-case equilibria, in Meinel, C. and Tison, S. (ed.) Lecture Notes in Computer Science, Springer Berlin Heidelberg.

Kumar, P.R. and T.-H. Shiau (1981). Existence of value and randomized strategies in zero-sum discrete-time stochastic dynamic games, SIAM Journal on Control and Optimisation, vol. 19, no. 5, 617-634.

Luss, H. (1999) 'On equitable resource allocation problems: A lexicographic minimax approach', Operations Research, vol. 47, no. 3, pp. 361-378.

Maitra, A. and T. Parthasarathy (1970). On stochastic games, Journal of Optimisation Theory and Applications, vol. 5, no. 4, 289-300.

Salmon, D. (1968) 'Minimax controller design', IEEE Transactions on Automatic Control, pp. 369-376.

Satayapiwat, P., K. Suksomboon, and C. Aswakul (2008). Reliability Evaluation by Expected Achievable Capacity in Stochastic Network Using Game Theory, International Conference on Telecommunications, 1 - 6.

Seo, D., A. Ali, , W.-T. Lim, N. Rafique and M. Thottethodi (2005). Near-optimal worst-case throughput routings for two-dimensional mesh networks, Proceedings of the 32nd annual international symposium on Computer Architecture, 432-443.