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Continuum modeling of park-and-ride services with linear complementarity system approach

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ABSTRACT

This paper studies the modeling of multimodal choice in a railway/highway system with continuum park-and-ride services along a linear traffic corridor. Heterogeneous commuters choose travel mode among auto, railway and park-and-ride. Both the traffic congestion on highway and crowding effects on rail transit are considered. The highway capacity is assumed to be stochastic to take into account the travel time reliability on highway. Commuters are assumed to be distributed continuously along the corridor. A linear complementarity system is proposed to model the modal choice for heterogeneous commuters along the corridor and solve the spatial equilibrium pattern. The formulated linear complementarity system is transformed into a mixed integer linear program to be solved. The modeling approach and solution algorithm are implemented in a small numerical example. The resultant solutions show that direct rail mode is chosen by commuters living close to the CBD and park-and-ride is chosen by those far away from the city center, while in the middle auto mode is preferred. Furthermore, when the stochastic link capacity is considered, more commuters choose to use park-and-ride mode for higher reliability.

Keywords: Park-and-ride service, Multimodal choice, Linear complementarity system, Travel time reliability, Heterogeneous commuters

Introduction

As a useful travel demand management strategy, park-and-ride services (P&R) have been widely used since 1930s (Noel 1988; Foo Tuan 1997; Lam et al. 2001; Hounsell et al. 2011). The principle of P&R service is to encourage commuters to choose combined auto and public transit travel mode to reduce auto traffic in the CBD area.

Most of the previous studies are restricted to the discrete network modeling approaches, but the continuum modeling approaches to transportation models are now gaining much more attention because of their advantages in dealing with dense-network models, macroscopic problems, and initial phase planning (Ho and Wong 2006). Wang et al. (2003) formulated a multimodal user equilibrium (UE) model for the city in the morning rush hours to determine the optimal P&R location and parking charge with the objectives of profit maximization and social cost minimization. Wang et al. (2004) considered a congested highway and a congestion-free railway to compare the characteristics of the modal choice equilibrium before and after a P&R service is introduced, and derived the optimal P&R locations and parking charges to maximize profit and minimize social cost. As an extension to Wang et al. (2004), Liu et al. (2009) relaxed two assumptions: railway is congestion-free and only one P&R facility is considered. A deterministic continuum equilibrium model is developed to represent modal choice and P&R transfer behaviors, and the model is transformed to a discrete problem by a supernetwork approach in the end. Although Liu et al. (2009) proposed an infinite mathematical programming method to formulate the similar spatial equilibrium travel pattern problem, the optimization formulation of the UE modal choice is not applicable for modeling multi-class equilibrium with heterogeneous commuters.

This study contributes to the literature in two major aspects. Firstly, linear complementarity system (LCS), which belongs to a class of modeling paradigm of differential variational inequalities (DVI) (Pang and Stewart 2008), is adopted to formulate the multimodal choice behavior throughout the corridor with heterogeneous commuters. The multimodal choice on each home location is modeled into a complementarity problem, while the spatial interaction of the travel time among different locations can be characterized by ordinary differential equations (ODEs). Therefore, it is natural to apply the DVI paradigm to model the spatial equilibrium travel pattern with multimodal choice in a linear traffic corridor with continuum P&R services. Secondly, the travel time reliability is considered in modeling the multimodal choice behavior. We assume stochastic highway capacity and hence stochastic travel time for highway users. Commuters using highway have to reserve a time budget to make sure not being late with certain level of punctuality requirement. Higher level of travel time reliability is regarded as one of the advantages in favor of rail transit usage and taking into account this factor makes the model more realistic.

Preliminary assumptions

A corridor with continuous entry points to the highway, railway, and P&R transfer services is considered, and CBD is assumed to be the destination and only exit point for all the commuters, as is shown in Figure 1.

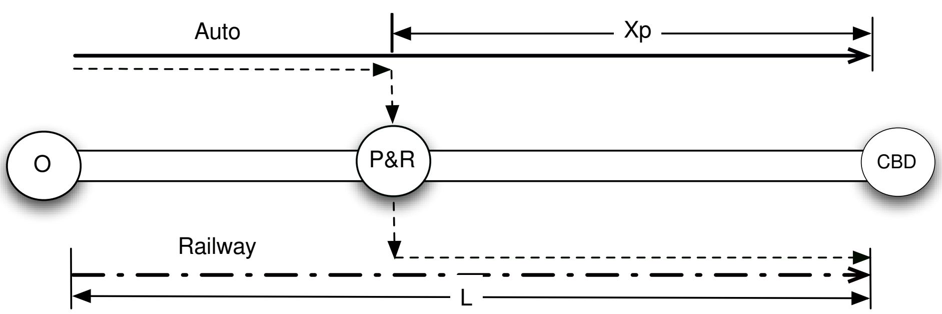


Figure 1 - A corridor with multimodal choice

With continuum modeling approach applied, it is assumed that both the entry points and P&R interchange services are infinite along the corridor, and thereafter, commuters can choose among three modes: auto, railway and P&R at any entry point to go to CBD. P&R users can drive private car from home and transfer to railway at any location to complete the trips. Let L represent the length of the corridor, x_p denote the P&R facility location, which is the distance from the location to CBD. The P&R facilities are assumed to be distributed continuously along the corridor. It is also assumed that commuters are continuously distributed along the corridor with travel demand density of $q_0(x)$ at location x, wherein x is the distance from the location to CBD.

Furthermore, heterogeneity is considered in this paper. It is assumed that heterogeneous commuters have different punctuality requirements when considering travel time reliability. It is also well known that travelers may value travel time differently, depending on the income level or trip characteristics, and thus heterogeneous commuters with different value of time have to be considered.

We denote $q_{m,n}^h(x)$ and $q_{m,n}^r(x)$ as the travel demand density (number of travelers per unit distance) for users choosing auto and rail mode at location x respectively, and $q_{m,n}^p(x, x_p)$ as the demand density of P&R users driving onto highway at x and transfer at P&R location x_p to complete the rest part of the trip by train, wherein $\{m, n\}$ represents the commuter group with ρ^m and τ^n as the punctuality probability requirement and value of time respectively.

It should be noted that λ^m is corresponding to ρ^m to represent the heterogeneous risk aversion. It is also assumed that the total demand density for different commuter group $\{m, n\}$ at location x is given as $q_{m,n}^0(x)$, $m = 1,2, \ldots, M$, $n = 1,2, \ldots, N$, so we have

$$q_{m,n}^h(x) + q_{m,n}^r(x) + \int_0^x q_{m,n}^p(x, x_p) dx_p = q_{m,n}^0(x) \tag{1}$$

General travel cost along the corridor

General travel cost by auto

Travel time budget

When travel time on highway is considered as stochastic, travelers would reserve some additional time to avoid late arrival. As is done in Lo et al. (2006), travel time budget can be expressed as

$$T_b = E(T) + \lambda \sigma_T \tag{2}$$

where T_b is travel time budget, T is the travel time on highway, $E(T)$ and σ_T are the expectation and standard deviation of T respectively, λ is a parameter related to the reliability of punctual arrival ρ, which can be written as

$$P\{T \le T_b\} = P\{T \le E(T) + \lambda \sigma_T\} = \rho \tag{3}$$

Rearranging Eq.(3) to obtain:

$$P\left\{\frac{T-E(T)}{\sigma_T} \leq \lambda\right\} = \rho \tag{4}$$

Since $\frac{T-E(T)}{\sigma_T}$ is the standard normal variable, the cumulative distribution function $\emptyset(x)$ of the standard normal variable can be written as

$$\emptyset(\lambda) = \rho \tag{5}$$

Accordingly, larger λ leads to higher reliability ρ, and vice versa.

Stochastic link capacity

Events as unpredicted vehicle breakdowns, traffic signal failures and minor traffic accidents, may occur on the transportation network and result in reducing the effective link capacities and degrading the performance of the network (Lo and Tung 2003). Therefore, it is important to consider such impact into the transportation network modeling for design and planning purposes. It is further assumed that the link capacity C_x at location x is the single source of travel time uncertainty and the probability distribution of the link capacity at each location is known and given. For simplicity, uniform distribution is adopted here for illustration purpose. If the maximum capacity and the minimum capacity are denoted as C_{max} and C_{min} respectively, the uniform probability distribution function can be described as in Eq.(6).

$$f(C_x) = \frac{1}{C_{max} - C_{min}}, \qquad C_x \in [C_{min}, C_{max}] \tag{6}$$

where $C_{min} > 0$ and $C_{max} > 0$.

Let $t_h(v_h(x))$ represent the travel time driving unit distance at location x, where $v_h(x)$ is the traffic volume at location x. As is done in Glen (1987), the travel time is supposed to be fully described by a first-order model where $B = 1$.:

$$t_h\big(v_h(x)\big) = t_h^0\left[1 + A\left(\frac{v_h(x)}{C_x}\right)^B\right] \tag{7}$$

where t_h^0 is free flow travel time per unit of distance, A and B are parameters.

Assuming the link capacity is independent of the traffic volume on it (Lo and Tung 2003), we can derive the expectation and variance of travel time on one unit distance at location x.

$$E\left(t_h\big(v_h(x)\big)\right) = t_h^0 + t_h^0 A v_h(x) E\left({C_x}^{-1}\right) = t_h^0 + \frac{t_h^0 A v_h(x)(\ln C_{max} - \ln C_{min})}{C_{max} - C_{min}} \tag{8}$$

$$\begin{aligned} D\left(t_h\big(v_h(x)\big)\right) &= {t_h^0}^2 A^2 v_h(x)^2 D\left({C_x}^{-1}\right) = {t_h^0}^2 A^2 v_h(x)^2\left(E\left({C_x}^{-2}\right) - E\left({C_x}^{-1}\right)^2\right) \\ &= {t_h^0}^2 A^2 v_h(x)^2\left[\frac{1}{C_{max}C_{min}} - \frac{(\ln C_{max} - \ln C_{min})^2}{(C_{max} - C_{min})^2}\right] \end{aligned} \tag{9}$$

In order to analyze the travel time between any different locations on highway, we divide the highway into several unit sections and assume $T_h(x_i, x_j)$ is the travel time from section x_i to section x_j, then we can get

$$T_h(x_i, x_j) = \sum_{k=i}^{j} t_h\big(v_h(x_k)\big) \tag{10}$$

$$E\left(T_h(x_i,x_j)\right)=\sum_{k=i}^{j}E\left(t_h\left(v_h(x_k)\right)\right)=\sum_{k=i}^{j}\left[t_h^0+\frac{t_h^0 A v_h(x)(\ln C_{max}-\ln C_{min})}{C_{max}-C_{min}}\right] \tag{11}$$

$$D\left(T_h(x_i,x_j)\right)=\sum_{k=i}^{j}D\left(t_h\left(v_h(x_k)\right)\right)+2\sum_{a<b}Cov\left(t_h\left(v_h(x_a)\right),t_h\left(v_h(x_b)\right)\right),i\le a<b\le j \tag{12}$$

To calculate the covariance among the travel time on different link segments, we need to know covariance among the stochastic link capacities at different sections along the corridor, which can be obtained and calibrated from the real historical data and is assumed to be given in this study. In most of the previous studies, the distributions of link capacities are assumed to be independent, and this assumption should be reasonable for relatively few network disruptions in the form of traffic incidents, so the covariance part in Eq.(12) will be zero based on this assumption.

General travel cost

Since the traffic volume at location $\boldsymbol{x}$ on highway is the accumulation of travelers choosing auto mode from the boundary of the corridor to location $\boldsymbol{x}$ and the travelers choosing P&R with the entry points before location $\boldsymbol{x}$ and transfer points after location $\boldsymbol{x}$, therefore the total traffic volume of all commuter groups at location $\boldsymbol{x}$ on highway can be calculated as $\boldsymbol{v_h(x)}$.

$$v_h(x)=\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_x^L q_{m,n}^h(w)\,dw+\int_x^L\int_0^x q_{m,n}^p\left(w,w_p\right)dw_p dw\right) \tag{13}$$

The general travel cost by auto from location x to CBD for commuter group $\{m,n\}$ is shown as Eq.(14).

$$C_{m,n}^h(x)=\tau^n\left(t_h^{hh}+t_h^{pw}+T_{m,n}^b(x)\right)+f_h(x)+P_h^w \tag{14}$$

wherein τ^n is the value of time for commuter group $\{m,n\}$, t_h^{hh} represents the access time from home to highway, t_h^{pw} is the egress time from parking place to workplace, and $T_{m,n}^b(x)$ is the travel time budget for commuter group $\{m,n\}$ from location x to CBD. The operating cost is assumed to be linear function of the travel distance as $f_h(x)=f_h^0+\gamma x$, where f_h^0 is constant, which may include fixed part of the tolls, and γx presents the variable part of the operating cost from location x to CBD, such as the fuel, insurance and variable parts of the highway tolls, etc., and γ is the operating cost per unit distance of traveling. P_h^w denotes the parking fee at CBD.

General travel cost by railway

Similar to auto users, the total traffic volume of all commuter groups at location x by railway can be calculated as $v_r(x)$.

$$v_r(x)=\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_x^L q_{m,n}^r(w)\,dw+\int_x^L\int_x^w q_{m,n}^p\left(w,w_p\right)dw_p dw\right) \tag{15}$$

Let $g_r\left(v_r(x)\right)$ be the crowding cost on the train per unit distance at location x. As is done in Huang (2000), a simple linear function with respect to traffic volume on the train is applied to describe the crowding cost,

$$g_r(v_r(x)) = \alpha + \beta v_r(x) \tag{16}$$

where α, β are the calibrated parameters. Therefore, the crowding cost for commuters traveled from location x to CBD should be

$$G_r(x) = \int_0^x \left\{ \alpha + \beta \left[\sum_{m=1}^{M} \sum_{n=1}^{N} \left(\int_z^L q_{m,n}^r(w)\, dw + \int_z^L \int_z^w q_{m,n}^p(w, w_p)\, dw_p dw \right) \right] \right\} dz \tag{17}$$

The general travel cost by rail transit from location x to CBD is described as follows:

$$C_{m,n}^r(x) = \tau^n \left(t_r^{hr} + t_r^{rw} + \frac{x}{V_r} \right) + G_r(x) + f_r(x) \tag{18}$$

where $t_r^{hr} + t_r^{rw}$ represents the access and egress time, $\frac{x}{V_r}$ is the travel time from location x to CBD by train, wherein V_r is the average travel speed of the train. $f_r(x) = f_r^0 + \kappa x$ is assumed to describe the fare structure of railway services, where f_r^0 is the fixed part of the fare, and κ is the railway fare per unit distance.

General travel cost by P&R

The general travel cost by P&R is as below,

$$\begin{aligned} C_{m,n}^p(x, x_p) = {} & \tau^n t_h^{hh} + \tau^n \left(T_{m,n}^b(x) - T_{m,n}^b(x_p) \right) + f_h(x, x_p) + p_p(x_p) \\ & + \tau^n t_p^{pr} + u_p + \frac{x_p}{V_r} + G_r(x_p) + f_r(x_p) + \tau^n t_r^{rw} \end{aligned} \tag{19}$$

where $T_{m,n}^b(x) - T_{m,n}^b(x_p)$ and $f_h(x, x_p)$ are the travel time budget and operating cost from location x to P&R facility location x_p. $p_p(x_p)$ represents the parking fee at P&R facility location x_p, which can be calculated as $p_p(x_p) = p_h^w e^{-\frac{x_p^2}{2L}}$ and assumed to be given (Liu et al. 2009). t_p^{pr} is the access time from parking place to train station, and u_p is the inconvenience or disutility cost. $\frac{x_p}{V_r}$ is the travel time from P&R location x_p to CBD by train, $G_r(x_p)$ is the crowding cost from location x_p to CBD, and $f_r(x_p)$ the service fare from location x_p to CBD.

Continuum equilibrium model with multimodal choice

To present the continuum equilibrium model with multimodal choice, we first introduce some notations used throughout the paper.

Table 1 – List of notations and descriptions

Notation	Description
Indices	
i	Index of sections, $i \in \{1, \dots, v\}$
m	Index of different punctuality probability requirement, $m \in \{1, \dots, M\}$
n	Index of different value of time, $n \in \{1, \dots, N\}$
{*m*, *n*}	Index of commuter groups
Variables	
$q_{m,n}^{h,i}$	Travel demand density of commuter group $\{m, n\}$ choosing auto mode at section i
$q_{m,n}^{r,i}$	Travel demand density of commuter group $\{m, n\}$ choosing rail mode at section i
$q_{m,n}^{p,ij}$	Travel demand density of commuter group $\{m, n\}$ driving from section i to j and transferring to railway

$C_{m,n}^{*,i}$	Equilibrium cost for commuter group $\{m,n\}$ at section i
$T_{m,n}^{b,i}$	Travel time budget for commuter group $\{m,n\}$ at section i
$G^{r,i}$	Crowding cost on railway at section i
Parameters	
L	Corridor length (km)
v	Section number of the corridor
M	Number of different punctuality probability requirement
N	Number of different value of time
$q_{m,n}^{0,i}$	Total demand density of commuter group $\{m,n\}$ at section i (commuters/h/km)
τ^n	Time to cost coefficient for commuter group $\{m,n\}$ (S$/min)
ρ^m	Punctuality probability requirement for commuter group $\{m,n\}$
λ^m	Parameter related to the reliability of punctual arrival for commuter group $\{m,n\}$
A	BPR function parameter
t_h^0	BPR function parameter, free-flow travel time (min)
C_{max}	Maximal link capacity (veh/km)
C_{min}	Minimal link capacity (veh/km)
C	Deterministic link capacity (veh/km)
t_h^{hh}	Time from home to highway (min)
t_h^{pw}	Time from parking place to workplace (min)
f_h^0	Fixed part of the operating cost by auto (S$/veh)
γ	Operating cost per unit distance by auto (S$/km/veh)
p_h^w	Parking fee at workplace (S$/veh)
t_r^{hr}	Time from home to railway station (min)
t_r^{rw}	Time from railway station to workplace (min)
f_r^0	Fixed part of the railway fare (S$)
κ	Railway fare per unit distance (S$/km)
V_r	Average speed of the train (km/min)
α	Parameter of the crowding cost
β	Parameter of the crowding cost
t_h^{pr}	Time from parking place at P&R location to train station nearby (min)
u_p	Fixed transfer disutility cost (S$)

We model the UE conditions with complementarity formulation as following:

$$\begin{gathered}
0 \le q_{m,n}^h(x) \perp C_{m,n}^h(x) - C_{m,n}^*(x) \ge 0 \\
0 \le q_{m,n}^r(x) \perp C_{m,n}^r(x) - C_{m,n}^*(x) \ge 0 \\
0 \le q_{m,n}^p\left(x,x_p\right) \perp C_{m,n}^p\left(x,x_p\right) - C_{m,n}^*(x) \ge 0 \\
0 \le C_{m,n}^*(x) \perp q_{m,n}^h(x) + q_{m,n}^r(x) + \int_0^x q_{m,n}^p\left(x,w_p\right) dw_p - q_{m,n}^0(x) \ge 0 \\
x \in [0,L],\;\; x_p \in [0,x),\;\; m = 1,\dots,M,\;\; n = 1,\dots,N
\end{gathered} \tag{20}$$

where $C_{m,n}^*(x)$ represents the minimum general travel cost among the three modes at location x for commuter group $\{m,n\}$. The first three constraints in (20) make sure that at equilibrium, the individual travel cost by a mode at any location should be the minimum among the three modes if the mode is used at the location. The last complementarity constraint entails that total users by the three modes in group $\{m,n\}$ amounts to the given total travel demand density $q_{m,n}^0$ at location x.

LCS formulation

We seek to formulate the spatial equilibrium traffic assignment with multimodal choice into LCS. Firstly, we take second order derivative of travel time budget $T_{m,n}^{b}(x)$ on highway with respect to x (Appendix 1 provides the detailed derivation).

$$\frac{d^2T_{m,n}^{b}(x)}{dx^2} = -\left[\frac{t_h^0 A(\ln C_{max} - \ln C_{min})}{C_{max} - C_{min}} + \lambda^m t_h^0 A\sqrt{\frac{1}{C_{max}C_{min}} - \left(\frac{\ln C_{max} - \ln C_{min}}{C_{max} - C_{min}}\right)^2}\right] \cdot \sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^{h}(x) + \int_0^x q_{m,n}^{p}(x, w_p)\, dw_p - \int_x^L q_{m,n}^{p}(w, x)\, dw\right) \tag{21}$$

For simplicity, using $\boldsymbol{D}_{m,n}$ to denote the constant part in Eq.(21),

$$\boldsymbol{D}_{m,n} = \left[\frac{t_h^0 A(\ln C_{max} - \ln C_{min})}{C_{max} - C_{min}} + \lambda^m t_h^0 A\sqrt{\frac{1}{C_{max}C_{min}} - \left(\frac{\ln C_{max} - \ln C_{min}}{C_{max} - C_{min}}\right)^2}\right] \tag{22}$$

then we can get

$$\frac{d^2T_{m,n}^{b}(x)}{dx^2} = -\boldsymbol{D}_{m,n} \cdot \sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^{h}(x) + \int_0^x q_{m,n}^{p}(x, w_p)\, dw_p - \int_x^L q_{m,n}^{p}(w, x)\, dw\right) \tag{23}$$

Similarly, taking second order derivative of the crowding cost on railway (Appendix 2 provides the detailed derivation), then we can get

$$\frac{d^2G_r(x)}{dx^2} = -\beta \sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^{r}(x) + \int_x^L q_{m,n}^{p}(w, x)\, dw\right) \tag{24}$$

The above ODEs reflect the spatial interactions of the travel time and crowding cost among different locations respectively along the corridor. Furthermore, initial conditions and boundary conditions of the ODEs can be expressed as follows.

$$T_{m,n}^{b}(0) = 0, G_r(0) = 0 \tag{25}$$

$$\frac{dT_{m,n}^{b}}{dx}(x = L) = t_h^0,\ \frac{dG_r}{dx}(x = L) = \alpha \tag{26}$$

The initial conditions ensure that the travel time and crowding cost at city center will be zero. The boundary conditions entails that no travel demand will be generated beyond the corridor length L, i.e. $v_h(L) = v_r(L) = 0$.

Based on the above analysis, LCS can be used to describe the spatial equilibrium travel pattern with multimodal choice. The solution is to find $q_{m,n}^{h}(x)$, $q_{m,n}^{r}(x)$, $q_{m,n}^{p}(x, x_p)$, $C_{m,n}^{*}(x)$, $T_{m,n}^{b}(x)$, $G_r(x)$ so that the following conditions (A) and (B) are satisfied.

(A) For almost all $x \in (0, L]$

$$\frac{d^2T_{m,n}^b(x)}{dx^2} = -\boldsymbol{D}_{m,n} \cdot \sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^h(x) + \int_0^x q_{m,n}^p(x,w_p)\,dw_p - \int_x^L q_{m,n}^p(w,x)\,dw\right)$$

$$\frac{d^2G_r(x)}{dx^2} = -\beta\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^r(x) + \int_x^L q_{m,n}^p(w,x)\,dw\right)$$

$$0 \le q_{m,n}^h(x) \perp C_{m,n}^h(x) - C_{m,n}^*(x) \ge 0 \tag{27}$$

$$0 \le q_{m,n}^r(x) \perp C_{m,n}^r(x) - C_{m,n}^*(x) \ge 0$$

$$0 \le q_{m,n}^p(x,x_p) \perp C_{m,n}^p(x,x_p) - C_{m,n}^*(x) \ge 0$$

$$0 \le C_{m,n}^*(x) \perp q_{m,n}^h(x) + q_{m,n}^r(x) + \int_0^x q_{m,n}^p(x,w_p)\,dw_p - q_{m,n}^0(x) \ge 0$$

$$x \in [0,L],\ \ x_p \in [0,x),\ \ m = 1,\dots,M,\ \ n = 1,\dots,N$$

(B) The initial and boundary conditions:

$$T_{m,n}^b(0) = 0, G_r(0) = 0$$

$$\frac{dT_{m,n}^b}{dx}(x=L) = t_h^0,\ \ \frac{dG_r}{dx}(x=L) = \alpha \tag{28}$$

Discrete approximation of the continuum model

LCS formulated above is quite difficult to be solved due to the combination of complementarity conditions, ODEs and integral parts in the model. In this study, the numerical solution of the formulated LCS will be accomplished via a time-stepping scheme. Specifically, we divide the corridor length into equal length sub-intervals by a positive integer $v > 0$, so we get the length of each section $\varepsilon_v = \frac{L}{v}$. Then we obtain the numerical solution of the LCS by computing the discretized scheme: $\{q_{m,n}^{h,i}\}$, $\{q_{m,n}^{r,i}\}$, $\{q_{m,n}^{p,ij}\}$, $\{C_{m,n}^{*,i}\}$, $\{T_{m,n}^{b,i}\}$, $\{G^{r,i}\}$, $i = 1,2,\cdots,v$.

Note that $q_{m,n}^{p,ij}$ represents the demand density for the commuters in group $\{m,n\}$ choosing P&R combination mode, who get onto highway at section i and drive at least one section distance to transfer to transit at section j, which requires $1 \le j < i \le v$.

In the discretized scheme, the derivative can be approximated by the forward difference quotient:

$$\frac{d^2T_{m,n}^b(x)}{dx^2} \approx \frac{\left[T_{m,n}^b(x+\varepsilon_v) - T_{m,n}^b(x)\right] - \left[T_{m,n}^b(x) - T_{m,n}^b(x-\varepsilon_v)\right]}{\varepsilon_v^2} \tag{29}$$

$$\frac{d^2G_r(x)}{dx^2} \approx \frac{[G_r(x+\varepsilon_v) - G_r(x)] - [G_r(x) - G_r(x-\varepsilon_v)]}{\varepsilon_v^2} \tag{30}$$

Hence, the discretized approximation of the first two equations in condition (A) can be described as:

$$\frac{T_{m,n}^{b,i+1} + T_{m,n}^{b,i-1} - 2T_{m,n}^{b,i}}{\varepsilon_v^2} = -\boldsymbol{D}_{m,n}\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^{h,i} + \sum_{j=1}^{i-1} q_{m,n}^{p,ij} - \sum_{j=i+1}^{v} q_{m,n}^{p,ji}\right) \tag{31}$$

$$\frac{G^{r,i+1} + G^{r,i-1} - 2G^{r,i}}{\varepsilon_v^2} = -\beta\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^{r,i} + \sum_{j=i+1}^{v} q_{m,n}^{p,ji}\right) \tag{32}$$

Noting that $T_{m,n}^{b,i} > 0, G^{r,i} > 0, i = 1,2,\cdots,v$, the discretized equations can be converted into a standard form of complementarity problem, so we can obtain the following linear complementarity problem (LCP) in which the variables are $\left(q_{m,n}^{h,i}\right)_{i=1}^{v}, \left(q_{m,n}^{r,i}\right)_{i=1}^{v}, \left(q_{m,n}^{p,ij}\right)_{i=2}^{v}, \left(C_{m,n}^{*,i}\right)_{i=1}^{v}, \left(T_{m,n}^{b,i}\right)_{i=1}^{v}, \left(G^{r,i}\right)_{i=1}^{v}$.

$$
\begin{gathered}
0 \le T_{m,n}^{b,i} \perp T_{m,n}^{b,i+1} + T_{m,n}^{b,i-1} - 2T_{m,n}^{b,i} + \varepsilon_v^2 \boldsymbol{D}_{m,n} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(q_{m,n}^{h,i} + \sum_{j=1}^{i-1} q_{m,n}^{p,ij} - \sum_{j=i+1}^{v} q_{m,n}^{p,ji} \right) \ge 0 \\
0 \le G^{r,i} \perp G^{r,i+1} + G^{r,i-1} - 2G^{r,i} + \varepsilon_v^2 \beta \sum_{m=1}^{M} \sum_{n=1}^{N} \left(q_{m,n}^{r,i} + \sum_{j=i+1}^{v} q_{m,n}^{p,ji} \right) \ge 0 \\
0 \le q_{m,n}^{h,i} \perp (\tau^n t_h^{hh} + \tau^n t_h^{pw} + \tau^n T_{m,n}^{b,i} + f_h^0 + \gamma i \varepsilon_v + P_h^w - C_{m,n}^{*,i}) \ge 0 \\
0 \le q_{m,n}^{r,i} \perp (\tau^n t_r^{hr} + \tau^n t_r^{rw} + \tau^n \frac{i\varepsilon_v}{V_r} + G^{r,i} + f_r^0 + \kappa i \varepsilon_v - C_{m,n}^{*,i}) \ge 0 \\
0 \le q_{m,n}^{p,ij} \perp \begin{pmatrix} \tau^n t_h^{hh} + \tau^n \left(T_{m,n}^{b,i} - T_{m,n}^{b,j} \right) + f_h^0 + \gamma (i-j) \varepsilon_v + p_h^w e^{-\frac{j^2}{2v}} + \tau^n t_p^{pr} \\ + u_p + \tau^n \frac{j\varepsilon_v}{V_r} + G^{r,i} + f_r^0 + \kappa j \varepsilon_v + \tau^n t_r^{rw} - C_{m,n}^{*,i} \end{pmatrix} \ge 0 \\
0 \le C_{m,n}^{*,i} \perp q_{m,n}^{h,i} + q_{m,n}^{r,i} + \sum_{j=1}^{i-1} q_{m,n}^{p,ij} - q_{m,n}^{0,i} \ge 0 \\
i = 1, \ldots, v, \;\; j = 1, \ldots, i, \;\; m = 1, \ldots, M, \;\; n = 1, \ldots, N
\end{gathered} \tag{33}
$$

The initial and boundary conditions (B) can be rewritten in the following discretized form:

$$
\begin{gathered}
T_{m,n}^{b,0} = 0, G^{r,0} = 0 \\
T_{m,n}^{b,v+1} - T_{m,n}^{b,v} = t_h^0 \varepsilon_v \, , G^{r,v+1} - G^{r,v} = \alpha \varepsilon_v
\end{gathered} \tag{34}
$$

Eqs. (34) can be substituted into (33) directly to simplify the calculation.

Transformation of LCP to Mixed Integer Linear Program

In this study, an equivalent mixed integer linear program (MILP) method is used to reformulate the LCP. Firstly, bilinear formulation approach is used to state the LCP as a mixed integer bilinear program (MIBLP) as follows:

$$
\begin{gathered}
\min y^T (Mx + q) + (e - y)^T x \\
s.t. \;\; x \ge 0, Mx + q \ge 0 \\
y \;\; binary
\end{gathered} \tag{35}
$$

where e presents the vector of n ones.

Proposition 1. *The solution of the proposed LCP (33) is equivalent to the solution of the MIBLP problem (35) with zero as minimal objective value.*

Proof. If x solves original LCP, define

$$
y_i = \begin{cases} 0, & if \; M_i x + q_i > 0 \\ 1, & if \; x_i > 0 \end{cases} \tag{36}
$$

then the x solving (33) render zero as objective value of MIBLP problem (35). If x solves (35) with zero as minimal objective value, then $y^T(Mx + q) + (e - y)^T x = 0$. Since y is binary, so either y^T or $(e - y)^T$ must be zero and another one must be one. In that case, there must be one of $Mx + q$ and x to be zero so that $y^T(Mx+q)+(e-y)^T x = 0$, then it also fulfills that $x^T(Mx+q) = 0$, which is the solution to LCP (33) as well.

Let M_{ij} denotes the (i,j)th element of M, (35) can be rewritten as follows:

$$\begin{aligned} &\min\ q^T y + \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij} y_i x_j + (e-y)^T x \\ &s.t.\ \sum_{j=1}^{n} M_{kj} x_j + q_k \geq 0, \qquad k = 1,2,\cdots,n \\ &\qquad x \geq 0, \qquad y\ binary \end{aligned} \tag{37}$$

Finally, we apply reformulation-linearization technique (RLT) to covert the MIBLP into a MILP, and the proof of equivalence can be referred to Sherali et al. (1998). This method transforms the problem into a higher dimensional space problem such that its continuous relaxation approximates the closure of the convex hull of feasible solutions to the underlying mixed integer program (MIP) problem. Specifically, multiply constrains in (35) by $(1-y_i)$ and y_i to reformulate the problem, and then linearize it by substituting $w_{ij} = y_i x_j$. Therefore, we can get the equivalent MILP as follows:

$$\begin{aligned} &\min\ q^T y + \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij} w_{ij} \\ &s.t.\ \sum_{j=1}^{n} M_{kj} w_{ij} + q_k y_i \geq 0, \qquad \forall\,(i,k) \\ &\sum_{j=1}^{n} M_{kj} x_j + q_k \geq \sum_{j=1}^{n} M_{kj} w_{ij} + q_k y_i, \qquad \forall\,(i,k) \\ &0 \leq w_{ij} \leq x_j, \forall\,(i,j), \qquad with\ \ w_{jj} = x_j, \forall j \\ &\qquad x \geq 0, \qquad y\ binary \end{aligned} \tag{38}$$

Numerical results

In this section, a set of numerical results is presented in an example corridor. The corridor is divided into v sections, and the deterministic UE modal choice is achieved on each section. On each section the link capacity is stochastic and follows uniform distribution. Here we analyze a single user class problem with the following parameter values: $L = 20$, $v = 10$, $\tau = 0.6$, $A = 0.6$, $t_h^0 = 1$, $C_{max} = 14000$, $C_{min} = 4000$, $C = 9000$, $t_h^{hh} = 2$, $t_h^{pw} = 3$, $f_h^0 = 2$, $\gamma = 0.02$, $p_h^w = 8$, $t_r^{hr} = 11$, $t_r^{rw} = 5$, $f_r^0 = 0.3$, $\kappa = 0.06$, $V_r = 0.8$, $\alpha = 0.002$, $\beta = 0.00003$, $t_h^{pr} = 1$, $u_p = 0.5$ and $q^{0,i} = 800$ for $i = 1,2,\ldots,10$.

The corridor is divided into 10 sections, and sections from 1 to 10 represent the locations from CBD to the boundary of the corridor. First of all, we examine the modal split pattern at equilibrium for the scenario considering travel time reliability with punctuality requirement $\rho = 95\%$.

Table 2 – Modal split pattern at equilibrium status with stochastic link capacity (commuters/h/km)

Section	Auto user density	Railway user density	P&R user density
1	0	800	0
2	0	800	0
3	0	800	0
4	0	800	0
5	800	0	0
6	492.9	0	307.1
7	0	0	800
8	0	0	800
9	0	0	800
10	0	0	800

The modal split pattern at equilibrium with stochastic link capacity is illustrated in Table 2. Rail mode is chosen at locations close to CBD, where the traffic congestion on highway is tremendous. On the contrary, both auto mode and P&R are used far away from CBD where highway traffic congestion is light. Specifically, all users choose railway from location 1 to 4, and auto is used at location 5 and 6, where the transfer cost of P&R mode is too high to be offset and commuters prefer direct auto mode. At locations that are close to city boundary, from location 7 to 10, P&R is the best choice, where commuters drive the car in a relatively light traffic condition on the highway that are far from the city center and then take advantage of public transit to avoid traffic congestion around the city center.

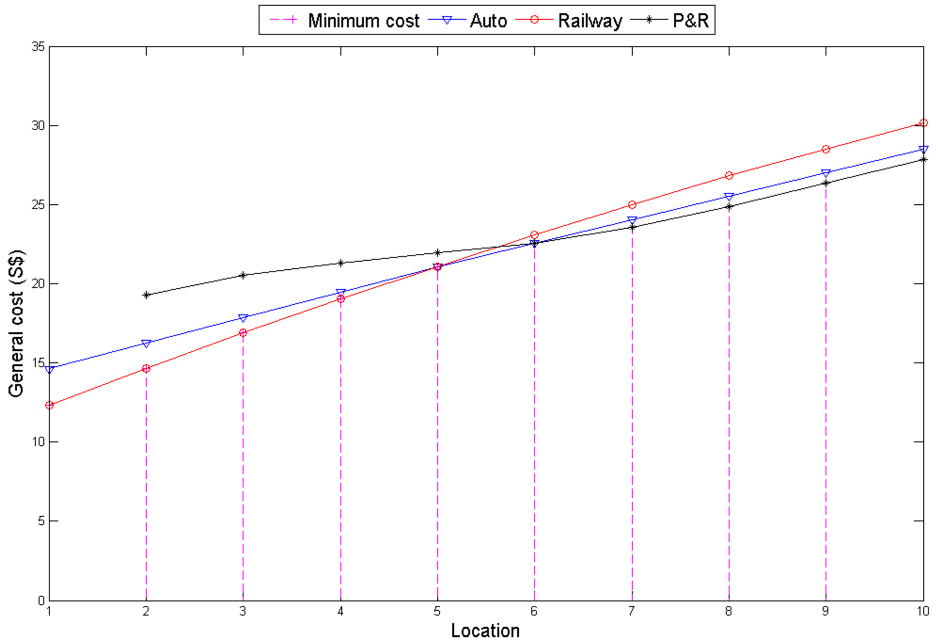


Figure 2 - General cost of three modes at equilibrium status with stochastic link capacity

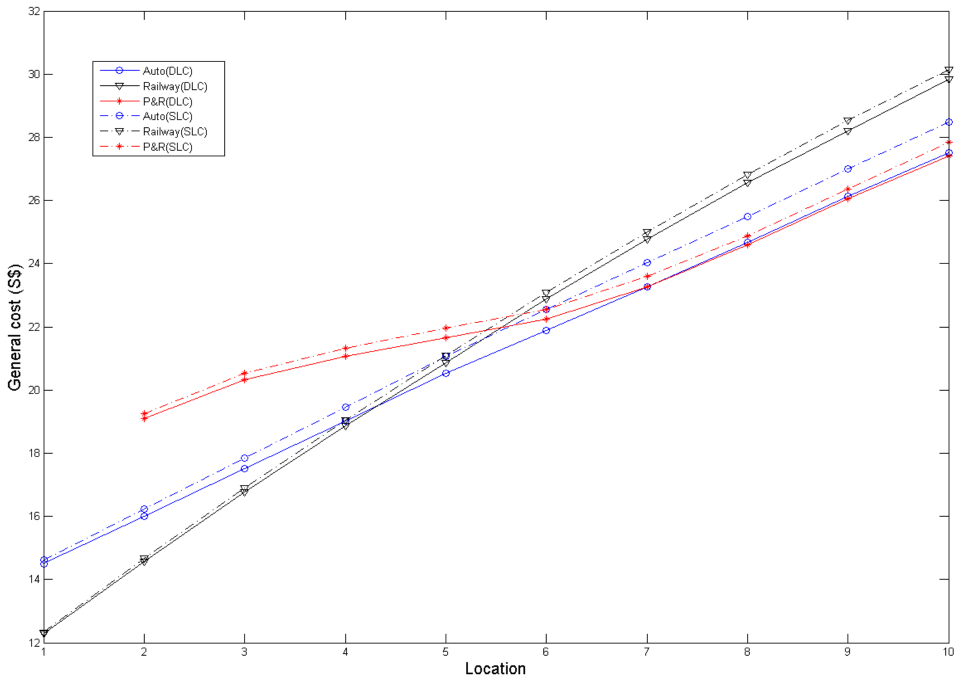
Figure 2 depicts the general cost by three modes along the corridor. From CBD to the boundary of the corridor, the general cost of railway is less than other modes at the first half corridor, and then it takes lowest general cost by auto mode from location 5 to 6. From location 7 to the corridor boundary, P&R is preferred according to its lowest general cost. The general cost of P&R at CBD (location 1) is not available since no one will choose P&R at destination as assumed in the previous content.

Table 3 – Comparison of mode split patterns between DLC and SLC (commuters/h/km)

Section	DLC			SLC ($\rho = 95\%$)		
	Auto user density	Railway user density	P&R user density	Auto user density	Railway user density	P&R user density
1	0	800	0	0	800	0
2	0	800	0	0	800	0
3	0	800	0	0	800	0
4	0	800	0	0	800	0
5	800	0	0	800	0	0
6	800	0	0	492.9	0	307.1
7	49	0	751	0	0	800
8	0	0	800	0	0	800
9	0	0	800	0	0	800
10	0	0	800	0	0	800

DLC - deterministic link capacity, *SLC* - stochastic link capacity

Table 3 compares the modal split patterns between the two scenarios of with and without considering the stochastic link capacity. As is shown in table 3, the resultant modal choices are the same from location 1 to 4 and location 8 to 10 respectively, which are the two extreme cases that are close and far away from the city center. In the middle of the city area, location 5 to 7, more commuters choose P&R when considering stochastic link capacity. The reasons are obvious as more travel time needs to be reserved on highway to take into account the travel time reliability, thus P&R is more favorable.



DLC - deterministic link capacity, *SLC* - stochastic link capacity

Figure 3 - Comparison of general costs at equilibrium status between DLC and SLC

Figure 3 illustrates the different general costs before and after considering stochastic link capacity. It is obvious that all the general costs of three modes increase after considering stochastic link capacity because people should consider travel time budget for higher reliability, which results in changing the modal split pattern, and more people choose railway.

Finally, the comparison of mode split results with different travel time reliability requirements are made for sensitivity analysis.

Table 4 – Comparison of mode split patterns with stochastic link capacity among different reliability requirements (commuters/h/km)

Section	$\rho = 95\%$			$\rho = 80\%$			$\rho = 70\%$		
	Auto user density	Railway user density	P&R user density	Auto user density	Railway user density	P&R user density	Auto user density	Railway user density	P&R user density
1	0	800	0	0	800	0	0	800	0
2	0	800	0	0	800	0	0	800	0
3	0	800	0	0	800	0	0	800	0
4	0	800	0	0	800	0	0	800	0
5	800	0	0	800	0	0	800	0	0
6	492.9	0	307.1	715.6	0	84.4	782.3	0	17.7
7	0	0	800	0	0	800	0	0	800
8	0	0	800	0	0	800	0	0	800
9	0	0	800	0	0	800	0	0	800
10	0	0	800	0	0	800	0	0	800

Table 4 shows the modal split patterns at equilibrium status with different travel time reliability requirements, it shows that the number of auto users decreases in terms of the growing of reliability requirement, while the amount of P&R mode users increases. It is easy to understand this situation since higher reliability requires more travel time budget and higher general cost by auto, in this case some commuters choose rail transit or P&R mode instead.

To learn how crowding effect on railway influences the mode split pattern, different parameter values of the crowding cost function in Eq.(16) are used and the comparisons of the results are shown in Table 5.

Table 5 – Comparison of mode split patterns with different parameter values of crowding effect on railway (commuters/h/km)

Section	$\alpha = 0.003$, $\beta = 0.0001$			$\alpha = 0.002$, $\beta = 0.00003$			$\alpha = 0.001$, $\beta = 0.00001$		
	Auto user density	Railway user density	P&R user density	Auto user density	Railway user density	P&R user density	Auto user density	Railway user density	P&R user density
1	0	800	0	0	800	0	0	800	0
2	0	800	0	0	800	0	0	800	0
3	0	800	0	0	800	0	0	800	0
4	576.2	223.8	0	0	800	0	0	800	0
5	800	0	00	800	0	0	54.5	745.5	0
6	800	0	0	492.9	0	307.1	0	0	800
7	502.9	0	297.1	0	0	800	0	0	800
8	698	0	102	0	0	800	0	0	800
9	0	0	800	0	0	800	0	0	800
10	0	0	800	0	0	800	0	0	800

From Table 5, one can find that, as the parameter values in the crowding cost function decrease, crowding cost goes down and more people choose rail and P&R modes while less people choose auto mode.

Conclusions

In this paper, the spatial equilibrium travel pattern with multimodal choice is formulated into LCS, and then it is transformed into an equivalent MILP by a discrete approximation method. A numerical example is conducted to compare the equilibrium patterns before and after the stochastic link capacity is considered, and the commuters' modal choice and P&R behaviors are characterized and analyzed. Numerical results show that public transit is preferred by users living close to CBD, while P&R mode is the prior choice for commuters far from CBD, and auto is also used around the middle part of the corridor, where the general cost of P&R is higher than direct auto mode in terms of the high transfer cost of P&R. In addition, when considering higher travel time reliability, more commuters will transfer from auto mode to P&R since the increase of general cost is so little that it can be ignored compared to the large increase of reliability.

It is proved in this study to be a successful attempt to apply the DVI modeling approach to model spatial equilibrium travel pattern with multimodal choice in a metropolitan area. However, only static equilibrium pattern is described. In the future study, dynamic elements will be introduced to model a dynamic spatial equilibrium with multiple transportation modes in a traffic corridor.

Appendix 1. Deriving the second order derivative of $T_{m,n}^{b}(x)$

Since the travel time function is continuous and partial derivable with respect to x, we can get the second order derivative of the travel time budget on highway.

The expectation of travel time $T_h(x)$ from location x to CBD is as below,

$$E\big(T_h(x)\big) = \int_{C_{min}}^{C_{max}} T_h(x) f(C_x)\, dx$$
$$= \int_{C_{min}}^{C_{max}} \int_0^x t_h^0 \left\{ 1 + A \left[\frac{\sum_{m=1}^{M} \sum_{n=1}^{N} \left(\int_z^L q_{m,n}^h(w)\, dw + \int_z^L \int_0^z q_{m,n}^p\big(w, w_p\big) dw_p dw \right)}{y} \right] \right\} \quad (39)$$
$$dz \frac{1}{C_{max} - C_{min}} dy$$

According to Theorem 1 stated below, the derivative of $E\big(T_h(x)\big)$ can be derived.

Theorem 1. *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t) dt, \qquad a \le x \le b \quad (40)$$

is continuous on $[a, b]$, and differentiable on (a, b), and $g'(x) = f(x)$.

In this study, the travel time function is continuous, taking first order derivative of the expectation function, we get

$$\frac{dE\left(T_h(x)\right)}{dx}=$$
$$\int_{C_{min}}^{C_{max}} t_h^0\left\{1+A\left[\frac{\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_x^L q_{m,n}^h(w)\,dw+\int_x^L\int_0^x q_{m,n}^p\left(w,w_p\right)dw_p dw\right)}{y}\right]\right\}\frac{1}{C_{max}-C_{min}}dy \tag{41}$$

As for the double integral $\int_x^L\int_0^x q_{m,n}^p\left(w,w_p\right)dw_p dw$, taking first order derivative of it, then we get

$$\frac{d\left(\int_x^L\int_0^x q_{m,n}^p\left(w,w_p\right)dw_p dw\right)}{dx}=-\int_0^x q_p\left(x,w_p\right)dw_p+\int_x^L q_p(w,x)\,dw \tag{42}$$

Then the second order derivative of the expectation function can be derived.

$$\frac{d^2E\left(T_h(x)\right)}{dx^2}$$
$$=\int_{C_{min}}^{C_{max}}\frac{t_h^0 A}{C_{max}-C_{min}}\left[\frac{\sum_{m=1}^{M}\sum_{n=1}^{N}\left(-q_{m,n}^h(x)-\int_0^x q_{m,n}^p\left(x,w_p\right)dw_p+\int_x^L q_{m,n}^p(w,x)\,dw\right)}{y}\right]dy \tag{43}$$
$$=-\frac{t_h^0 A(\ln C_{max}-\ln C_{min})}{C_{max}-C_{min}}\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^h(x)+\int_0^x q_{m,n}^p\left(x,w_p\right)dw_p-\int_x^L q_{m,n}^p(w,x)\,dw\right)$$

Similarly, we can get the second order derivative of the standard deviation function.

$$\sigma\left(T_h(x)\right)=\sqrt{D\left(\int_0^x t_h^0\left\{1+A\left[\frac{\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_z^L q_{m,n}^h(w)\,dw+\int_z^L\int_0^z q_{m,n}^p\left(w,w_p\right)dw_p dw\right)}{C_z}\right]\right\}dz\right)}$$
$$=\sqrt{D\left({C_z}^{-1}\int_0^x t_h^0 A\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_z^L q_{m,n}^h(w)\,dw+\int_z^L\int_0^z q_{m,n}^p\left(w,w_p\right)dw_p dw\right)dz\right)}$$
$$=\sqrt{\left(\int_0^x t_h^0 A\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_z^L q_{m,n}^h(w)\,dw+\int_z^L\int_0^z q_{m,n}^p\left(w,w_p\right)dw_p dw\right)dz\right)^2 D\left({C_z}^{-1}\right)} \tag{44}$$
$$=\int_0^x t_h^0 A\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_z^L q_{m,n}^h(w)\,dw+\int_z^L\int_0^z q_{m,n}^p\left(w,w_p\right)dw_p dw\right)dz$$
$$\cdot\sqrt{\frac{1}{C_{max}C_{min}}-\left(\frac{\ln C_{max}-\ln C_{min}}{C_{max}-C_{min}}\right)^2}$$

Taking second order derivative of Eq.(44), we get

$$\frac{d^2\sigma\left(T_h(x)\right)}{dx^2}=-t_h^0 A\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^h(x)+\int_0^x q_{m,n}^p\left(x,w_p\right)dw_p-\int_x^L q_{m,n}^p(w,x)\,dw\right)\cdot\sqrt{\frac{1}{C_{max}C_{min}}-\left(\frac{\ln C_{max}-\ln C_{min}}{C_{max}-C_{min}}\right)^2} \tag{45}$$

Finally we can get the second order derivative of the travel time budget function as follows:

$$\frac{d^2T_{m,n}^b(x)}{dx^2}=\frac{d^2E\big(T_h(x)\big)}{dx^2}+\lambda^{\mathrm{m}}\frac{d^2\sigma\big(T_h(x)\big)}{dx^2}$$
$$=-\frac{t_h^0A(\ln C_{max}-\ln C_{min})}{C_{max}-C_{min}}\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^h(x)+\int_0^x q_{m,n}^p\big(x,w_p\big)\,dw_p-\int_x^L q_{m,n}^p(w,x)\,dw\right)$$
$$-\lambda^{\mathrm{m}}t_h^0A\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^h(x)+\int_0^x q_{m,n}^p\big(x,w_p\big)\,dw_p-\int_x^L q_{m,n}^p(w,x)\,dw\right)\cdot$$
$$\sqrt{\frac{1}{C_{max}C_{min}}-\left(\frac{\ln C_{max}-\ln C_{min}}{C_{max}-C_{min}}\right)^2} \tag{46}$$
$$=-\left[\frac{t_h^0A(\ln C_{max}-\ln C_{min})}{C_{max}-C_{min}}+\lambda^m t_h^0A\sqrt{\frac{1}{C_{max}C_{min}}-\left(\frac{\ln C_{max}-\ln C_{min}}{C_{max}-C_{min}}\right)^2}\right]\cdot$$
$$\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^h(x)+\int_0^x q_{m,n}^p\big(x,w_p\big)\,dw_p-\int_x^L q_{m,n}^p(w,x)\,dw\right)$$

Appendix 2. Deriving the second order derivative of $\boldsymbol{G_r(x)}$

As for the crowding cost on railway

$$G_r(x)=\int_0^x\left[\alpha+\beta\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_z^L q_{m,n}^r(w)\,dw+\int_z^L\int_z^w q_{m,n}^p\big(w,w_p\big)\,dw_p dw\right)\right]dz \tag{47}$$

According to Theorem 1, the first order derivative of $G_r(x)$ can be derived.

$$\frac{dG_r(x)}{dx}=\alpha+\beta\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\int_x^L q_{m,n}^r(w)\,dw+\int_x^L\int_x^w q_{m,n}^p\big(w,w_p\big)\,dw_p dw\right) \tag{48}$$

As for the double integral $\int_x^L\int_x^w q_{m,n}^p\big(w,w_p\big)\,dw_p dw$, taking first order derivative of it, such that

$$\frac{d\left(\int_x^L\int_x^w q_{m,n}^p\big(w,w_p\big)\,dw_p dw\right)}{dx}=-\int_x^L q_{m,n}^p(w,x)\,dw \tag{49}$$

Then the second order derivative of Eq.(49) can be derived as below.

$$\frac{d^2G_r(x)}{dx^2}=-\beta\sum_{m=1}^{M}\sum_{n=1}^{N}\left(q_{m,n}^r(x)+\int_x^L q_{m,n}^p(w,x)\,dw\right) \tag{50}$$

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