NUMERICAL SIMULATIONS OF A UNIFIABLE MULTI-COMMODITY KINEMATIC WAVE MODEL FOR TRAFFIC SYSTEMS WITH TRADABLE RIGHT-OF-WAY

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ABSTRACT

This study deals with the numerical simulations of traffic flow on a multi-lane road, where users with different values of time (VOT) can trade their rights-of-way (ROW). The resulting traffic flow violates the First-In-First-Out (FIFO) principle, since vehicles with higher VOTs would travel faster by paying those with lower VOTs. [7] presented a novel multi-commodity kinematic wave model for such a system based on 5 assumptions of unifiability, budget-balance, UE driver behavior, optimizing system cost, and benefit sharing among the users. The Riemann problem was analytically solved for a traffic stream with 3 commodities - with 2 groups of users with different values of time participating in the scheme, and a non-participating third group.

Here we numerically simulate the 3-commodity traffic stream on a road segment and compare the results with the analytical results. We use the unifiable multi-commodity Cell Transmission Model developed by [2]. Finally we conclude the study with a discussion of possible future extensions.

Key words: Unifiable multi-commodity kinematic wave model; Numerical Simulation; Value-of-time; Tradable rights-of-way; Riemann problem.

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1 Introduction

Real-world traffic data has shown that traffic does not follow FIFO (First In First Out) behavior [8]. In addition, there are many real-world mechanisms and applications that explicitly require FIFO violations, either for emergency purposes, such as ambulances, highway-assistance, law-enforcement and fire services, or to make more efficient use of infrastructure and resources, such as toll and HOT lanes(High-Occupancy Toll lanes), Bus rapid Transit (BRT) lanes, and so on.

Recently, motivated by the advances in connected vehicle technology, there has been an increased interest in exploring user-trading schemes that influence road users' choices, so as to manage traffic congestion. One example is the class of tradable credit schemes - [3], [4], [5]. Choice behaviors of people that have been explored in these schemes have ranged from route choice, mode choice, and departure time choice. Indeed, fundamentally new paradigms based on users trading their priorities of service have been proposed for efficient management of transportation systems in general [14]. These include ride-share systems [15] and signal control systems [16].

Recently, [7] presented the idea of vehicles with heterogeneous values-of-time (VOT) trading their rights-of-way (ROW) on a multi-lane road by negotiating their travel speeds, which seems to be one of the most basic choices that users can make. The study assumed the traffic stream to be non-FIFO and unifiable (explained in Section 2), and the users were assumed to show selfish cost-minimizing behavior. A 3-commodity system was considered (each commodity is a group of users with a VOT, which is different among different classes and same within the class). The system was optimized to minimize the total cost for all users, and unique optimum speed values were obtained for each group of users. The resulting benefits of the scheme in terms of the reduction in cost can be shared among the different users in several ways.

To consider the impact of such schemes, and to model traffic accurately even in the absence of such schemes, we need models that allow for FIFO violations so that we get a better idea of the traffic flow characteristics. One such multi-commodity unifiable kinematic wave model was developed in [2]. This model can be used to study different commodity shockwaves and total density shockwaves in traffic streams with heterogeneous groups of users, such as the one presented in [7]. [7] presented analytical solutions of the 3-commodity traffic system, and the Riemann problem was solved to see the total traffic and commodity shockwaves. Here, we are interested in numerical simulations of the total and commodity densities of the 3-commodity traffic system, where each commodity is a group of users, and the groups travel at different speeds. We will use the multi-commodity Cell Transmission Model, also presented in [2], to simulate our system.

The rest of the paper is arranged as follows: Section 2 deals with the description of the multi-commodity system and variables, and the results of the tradable ROW scheme for a 3-commodity system; Section 3 presents the Unifiable Multi-commodity Cell Transmission Model (CTM) scheme; Section 4 contains the description of the Riemann Problem, the 2 examples solved by the CTM, and the results of the simulation; the study is finally concluded and possible future extensions are provided in Section 5.

2 A Unifiable Multi-Commodity Kinematic Wave Model and Tradable Right of Way Scheme for a 3-Commodity System

2.1 A Unifiable Multi-commodity Traffic System

The total density on the road is $k(x,t)$, the average speed of traffic is $v(x,t)$, and the flow rate is $q(x,t)$. Let there be M commodities in the system. We can denote the density the density, average speed and flow-rate of commodity m (m = 1,...,M) by k_m, v_m and q_m respectively. Hereafter, (x,t) is omitted unless necessary. The following relations hold :

The commodity densities and flow-rates sum up to the total density and flow-rate, respectively,

$$\sum_{m=1}^{M} k_m = k \tag{1a}$$

$$\sum_{m=1}^{M} q_m = q \tag{1b}$$

The constitutive law holds for total traffic and for each commodity (m = 1,...,M),

$$q_m = k_m v_m \tag{2a}$$

$$q = kv \tag{2b}$$

The commodity and total traffic flow conservation equations are (m = 1,...,M):

$$\frac{\partial k_m}{\partial t} + \frac{\partial q_m}{\partial x} = 0, \tag{3a}$$

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{3b}$$

A multi-commodity traffic stream is said to be FIFO if all the commodities have the same speed, i.e.

$$v_m = v, \quad m = 1,2,...,M \tag{4a}$$

which is equivalent to:

$$\gamma_m(k,\vec{\mathbf{p}}) = 1, \quad m = 1,2,...,M \tag{4b}$$

Our model does not have FIFO as an explicit or implicit assumption.

Further, a unifiable traffic system is one in which the total traffic conditions depend only on the total density and not on the density proportions of different commodities. Mathematically,

$$v = V(k); \tag{5a}$$

$$q = k.V(k) \tag{5b}$$

We denote the commodity density proportions by

$$p_m = \frac{k_m}{k}, \quad m = 1,2,...,M \tag{6a}$$

where

$$\sum_{m=1}^{M} p_m = 1, \quad p_m \in [0,1], \quad m = 1,2,...,M \tag{6b}$$

The vector of commodity density proportions is denoted by

$$\vec{\mathbf{p}} = \sum_{m=1}^{M} p_m \vec{\mathbf{e}}_m \tag{6c}$$

where $\vec{\mathbf{e}}_m$ denotes the unit vector whose i^{th} element is 1.

The commodity speed proportions are denoted by

$$\gamma_m(k,\vec{\mathbf{p}}) = \frac{v_m}{v}, \tag{7a}$$

where

$$\gamma_m(k,\vec{\mathbf{p}}) > 0. \tag{7b}$$

The flow-rate proportions are denoted by

$$\xi_m(k,\vec{\mathbf{p}}) = \frac{q_m}{q} = p_m \cdot \gamma_m(k,\vec{\mathbf{p}}) \tag{8a}$$

which satisfies the following conditions :

$$\sum_{m=1}^{M} \xi_m(k,\vec{\mathbf{p}}) = 1, \quad \xi_m(k,\vec{\mathbf{p}}) \in [0,1]. \tag{8b}$$

Another useful way to look at the commodity speed proportions is looking at their relative speed ratios, $\beta_m(k,\vec{\mathbf{p}})$, written for brevity as β_m, :

$$\frac{v_1}{\beta_1} = \frac{v_m}{\beta_m} = \frac{v_M}{\beta_M}. \tag{9a}$$

which is equivalent to $(m,n = 1,2,...,M)$:

$$\frac{v_m}{v_n} = \frac{\beta_m}{\beta_n} \tag{9b}$$

And the weighted average of the relative speed ratios is denoted by β :

$$\beta = \sum_{m=1}^{M} p_m \beta_m \tag{9c}$$

Further, the absolute speed ratios can be defined in terms of the relative speed ratios as :

$$\gamma_m = \frac{\beta_m}{\beta} \tag{9d}$$

2.2 A Unifiable Multi-Comodity Kinematic Wave Model

We denote the density of commodity m by $k_m = p_m k$, which depends on both time and location. We denote the vector $\vec{\mathbf{p}} = (p_1, p_2)$, and $\beta_m = \sqrt{\pi_m}$ for $m = 1, 2$. Further we denote

$$\beta_3(\vec{\mathbf{p}}) = \frac{p_1\beta_1 + p_2\beta_2}{p_1 + p_2} = \frac{k_1\beta_1 + k_2\beta_2}{k_1 + k_2}, \tag{10}$$

$$\beta(\vec{\mathbf{p}}) = p_1\beta_1 + p_2\beta_2 + p_3\beta_3(\vec{\mathbf{p}}) = \beta_3(\vec{\mathbf{p}}). \tag{11}$$

Here $\beta_m(\vec{\mathbf{p}})$ are the relative speed ratios [9], among which the first two commodities' are constant, and the third commodity's depends on the commodity density proportions. $\beta(\vec{\mathbf{p}})$ is the average of the relative speed ratios.

Then we can easily show that

$$\frac{v_1}{\beta_1} = \frac{v_2}{\beta_2} = \frac{v_3}{\beta_3(\vec{\mathbf{p}})}, \tag{12}$$

and for $m = 1, 2, 3$

$$v_m = \frac{\beta_m(\vec{\mathbf{p}})}{\beta(\vec{\mathbf{p}})} V(k). \tag{13}$$

The kinematic wave model for a unifiable multi-commodity traffic system can be written as :

$$\frac{\partial k}{\partial t} + \frac{\partial k V(k)}{\partial x} = 0, \tag{14a}$$

$$\frac{\partial k_m}{\partial t} + \frac{\partial \frac{\beta_m k_m}{\beta(\vec{\mathbf{p}})} V(k)}{\partial x} = 0, \tag{14b}$$

2.3 Tradable Right of Way (TROW) Scheme for Users with Heterogeneous Values of Time (VOT)

We make the following assumptions about the system:

1. All or some vehicles can participate in trading their rights-of-way and changing their orders. (Choice)

2. The overall traffic stream is unchanged before and after the trade and the order change. In particular, those who do not participate are not disturbed or impacted. (Unifiability and Neutrality)

3. Different vehicles have different values of time. (Heterogeneity)

4. Vehicles have to pay a positive/negative price to travel faster/slower than the average traffic. (Prices)

5. The system has a balanced budget; i.e., the total price/credits exchanged among all users equals zero. It means that the transactions take place only among the users, and there is no third party involved. (Budget balance)

6. The objective of trading the right-of-way is to individually maximize utility/minimize cost. (Utility maximizers)

The simplest case for a multi-commodity system with tradable right-of-way is the two-commodity system, but it is not reasonable to assume total market penetration for such a system due to reasons such as user preferences, technological barriers, or just time-to-adoption of the technology. Here we assume that there are two groups of users, or commodities, which participate in trading their rights-of-way, who have VOTs at π_1 and π_2, with $\pi_1 < \pi_2$, and a third commodity with an unknown VOT which does not participate in the trade. Their proportions are p_1, p_2, and $1 - p_1 - p_2$ respectively, where

$$p_1 + p_2 < 1. \tag{15}$$

Essentially, the market penetration rate of the trade scheme is $p_1 + p_2$. The two participating commodities' prices/costs are c_1 and c_2, respectively. The two commodities can negotiate their respective speeds: v_1 and v_2, and the third commodity's speed is not impacted.

From the unifiability assumption, we have the full system fundamental diagram as

$$p_1 v_1 + p_2 v_2 + (1 - p_1 - p_2) V(k) \quad = \quad V(k), \tag{16}$$

which gives us

$$p_1 v_1 + p_2 v_2 \quad = \quad (p_1 + p_2) V(k). \tag{17}$$

From the balanced budget assumption, we have

$$p_1 c_1 + p_2 c_2 \quad = \quad 0, \tag{18}$$

Now for the commodity speeds, a commodity User Equilibrium (cUE) principle for users' speeds was presented in [7], and it says : Each commodity's cost with the chosen speed and price is less than or equal to the cost with the unchosen speed and price. For the purpose of simplicity, we also assume that all vehicles of the same commodity are coalescent; i.e., they always choose the same speed and price. One aspect to note here is that the cUE conditions are equivalent to the envy-free principle, common in economic literature and introduced in transportation literature in [14].

The minimization of the total system cost led to unique speed solutions for the commodity speeds :

$$v_1 \quad = \quad \frac{\sqrt{\pi_1}}{p_1\sqrt{\pi_1} + p_2\sqrt{\pi_2}}(p_1 + p_2)V(k), \tag{19a}$$

$$v_2 \quad = \quad \frac{\sqrt{\pi_2}}{p_1\sqrt{\pi_1} + p_2\sqrt{\pi_2}}(p_1 + p_2)V(k). \tag{19b}$$

The details and principles for determining the prices/exchanged credits between the commodities can be found in [7], and are not included here since they do not impact the traffic flow analysis here. Next we will describe the numerical scheme.

3 Unifiable Multi-Commodity Cell Transmission Model

To solve the kinematic wave model in (14), we will use the multi-commodity Cell Transmission Model (CTM) developed in [2].

The total traffic demand and supply are defined as :

$$\delta(k) \quad = \quad \phi(min\{k, K_c\}), \tag{20a}$$

$$\sigma(k) \quad = \quad \phi(max\{k, K_c\}). \tag{20b}$$

where K_c is the traffic critical density, corresponding to the maximum flow-rate.

In the unifiable multi-commodity CTM, the road segment is divided into cells of length Δx and the time duration into intervals with a step-size of Δt. The total density and the commodity density proportions for m = 1,2,...,M in cell i at time-step j are denoted by k_i^j and $p_{m,i}^j$ respectively. The corresponding commodity density is denoted by $k_{m,i}^j = k_i^j \; p_{m,i}^j$.

The total traffic demand and supply are calculated from (20) and the flow proportions $\xi_{m,i}^j$ are calculated from (8). The boundary fluxes for total and comodity traffic between cells $i-1$ and i can be calculated from the upstream cell's demand, the downstream cell's supply, and the upstream cell's commodity flow-proportions :

$$q_i^j \quad = \quad min\{\delta_{i-1}^j, \sigma_i^j\}, \tag{21a}$$

$$q_{m,i}^j \quad = \quad q_i^j . \xi_{m,i-1}^j. \tag{21b}$$

From the conservation of total and commodity traffic flows, we can update total density and density proportions as

$$
\begin{aligned}
k_i^{j+1} &= k_i^j + \frac{\Delta t}{\Delta x}.(q_i^j - q_{i+1}^j), && (22a)\\
k_{m,i}^{j+1} &= k_i^j.p_{m,i}^j + \frac{\Delta t}{\Delta x}.(q_{m,i}^j - q_{m,i+1}^j) && (22b)\\
p_{m,i}^{j+1} &= \frac{k_{m,i}^{j+1}}{k_i^{j+1}}. && (22c)
\end{aligned}
$$

Here, Δt and Δx should satisfy the following extended CFL condition [6]:

$$\frac{\Delta x}{\Delta t} \geq \max_{k\in[0,K]} |\lambda_1(k)| \cdot \max_{k\in[0,K],\vec{\mathbf{p}},m=1,2,\ldots,M} \gamma_m(k,\vec{\mathbf{p}}). \tag{23}$$

where $\lambda_1(k)$ is the characteristic wave speed for the total traffic. The proof for this requirement can be found in [wjin-2017-unifiable].

For our system :

$$\max_{k\in[0,K],\vec{\mathbf{p}},m=1,2,\ldots,M} \gamma_m(k,\vec{\mathbf{p}}) = \sqrt{\frac{\pi_2}{\pi_1}} \;\; (\pi_2 > \pi_1) \tag{24}$$

which can be verified easily from (19).

4 The Riemann Problem and the Numerical Examples

4.1 The Riemann Problem

The system of conservation laws, (14), is challenging to solve under general initial and boundary conditions. But we can solve the Riemann Problem under the following jump initial condition :

$$(k(x,0),\ \vec{\mathbf{p}}(x,0)) = \begin{cases} (k_L,\vec{\mathbf{p}}_L), & x < x_0; \\ (k_R,\vec{\mathbf{p}}_R), & x > x_0. \end{cases} \tag{25}$$

The analytical solutions for the total traffic and commodity waves for the Riemann problem were already presented in [7]. One thing to note here is that if $k_L < k_R$, the total traffic wave will be a shockwave, and if $k_L > k_R$, the total traffic wave will be a rarefaction wave.

4.2 The Numerical Examples

We take 3 commodities in the traffic system, with $\pi_1 = 1.0$, $\pi_2 = 2.0$, and the third non-participating commodity has an unknown VOT.

We assume the Greenshield's [11] speed-density relation to hold for the total traffic, with the jam density $K = 150\ veh/mile/lane$ and the freeflow speed, $v_f = 60\ mph$. Our example road system is a 2-laned 10-mile stretch of freeway, and we will look at the following 2 types of Riemann Problems (rarefaction wave and shockwave respectively):

$$(k(x,0),\ \vec{\mathbf{p}}(x,0)) \quad = \quad \begin{cases} (200,(0.2,0.4)), & x < 5; \\ (60,(0.3,0.2)), & x > 5\ . \end{cases} \tag{26a}$$

which corresponds to the case of shockwave in the total traffic, and :

$$(k(x,0),\ \vec{\mathbf{p}}(x,0)) \quad = \quad \begin{cases} (80,(0.4,0.3)), & x < 5; \\ (250,(0.35,0.25)), & x > 5\ . \end{cases} \tag{26b}$$

We simulate for a total time period of 1200 sec, with Δt of 6 sec. The following Δx is chosen, which satisfies the CFL condition (23) :

$$\Delta x \quad = \quad \Delta t\ .\ \max_{k\in[0,K]} |\lambda_1(k)|\ .\ \max_{k\in[0,K],\vec{\mathbf{p}},m=1,2,\ldots,M} \gamma_m(k,\vec{\mathbf{p}}). \tag{27}$$

For the Greenshield's Fundamental Diagram :

$$\max_{k\in[0,K]} |\lambda_1(k)| = v_f. \tag{28}$$

In the next subsection, we present the results of the simulations using the multi-commodity CTM.

4.3 Results from CTM

First lets look at the solution of the shockwave problem, (26b). The total density over the cells varies with time as shown in Figure 1, and the densities of commodities 1 and 2 vary as shown in Figure 2.

The solution of the rarefaction wave problem, (26a), is presented next. The total density varies as shown in Figure 3. And the densities of commodities 1 and 2 vary as shown in Figure 4.

As we can see from Figure 1, there is a clear backward traveling shockwave originating from the jump point. This is what the analytical solution of the Riemann problem suggests as well. These correspond to the 1-waves presented in [7]. From Figure 2, the shockwaves for the different commodities are seen, and these correspond to the 3-waves presented in [7]. These results are as expected, and show the applicability of the multi-commodity CTM to the traffic system with TROW presented in [7].

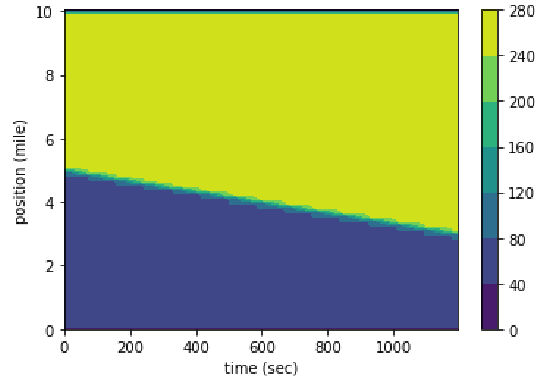
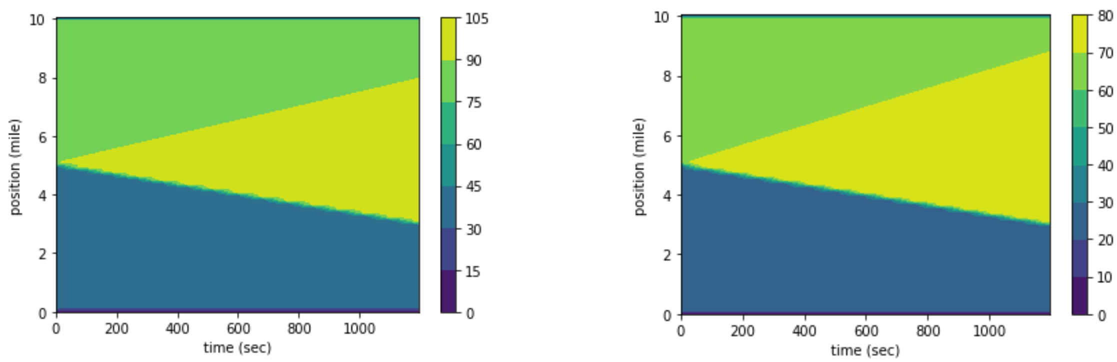


Figure 1: Total density shockwaves for the Riemann Problem



(a) Density of commodity 1

(b) Density of commodity 2

Figure 2: Commodity shockwaves in Riemann Problem

Next we look at the solution for the rarefaction wave Riemann Problem. The rarefaction waves are clearly seen in Figure 3 and Figure 4 for the total traffic as well as for each individual commodity. The results are in accordance with the expected rarefaction waves from the analytical model solutions, which were qualitatively presented in [7].

In the future, we would like to get the exact and not just qualitative analytical solutions and compare the closeness of the numerical solutions with the analytical solutions.

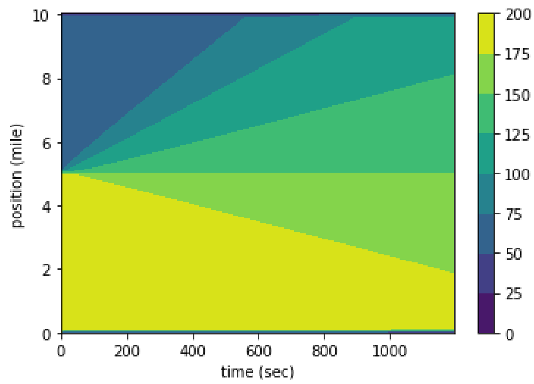
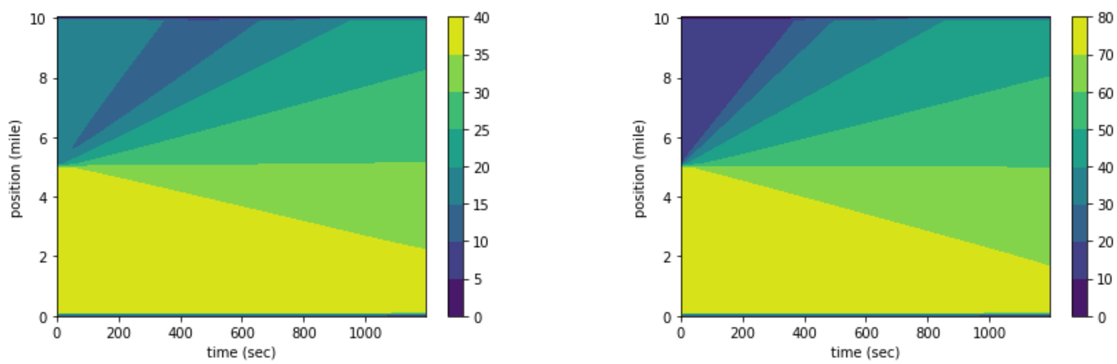


Figure 3: Total density rarefaction waves for the Riemann Problem



(a) Density of commodity 1

(b) Density of commodity 2

Figure 4: Commodity rarefaction waves in Riemann Problem

5 Conclusion and Future Work

Traffic streams have been known to violate the FIFO assumption, and thus it becomes important to have models that can replicate this phenomenon. In [7], a novel scheme was presented in which groups of users with different VOTs, called different commodities, negotiate their travel speeds on a multi-lane road. The unique results for the speeds that produce system optimum conditions while still satisfying each user's selfish behavior (commodity User Equilibrium) were included. We look at a simple case of a 3-commodity system - in which 2 groups of users exchange credits/money and negotiate their travel speeds, while the third group does not participate and is not impacted by the scheme. The framework for analyzing such a unifiable but non-FIFO multi-commodity system, presented in [2], was included here.

Next we looked at the unifiable multi-commodity Cell Transmission Model for such a system, also originally presented in [2]. We solved numerically the Riemann Problem for 2 cases : one for shockwave and one for rarefaction wave. The numerical results are qualitatively in accordance with the analytical solutions of the wave model. One thing that we would like to check in the future is the convergence of the CTM scheme, with different Δx and Δt values.

Some directions for extension of this study are :

1. Trying relationships other than the Greenshields', particularly the Triangular fundamental diagram.
2. Comparing quantitatively the analytical and numerical solutions for the Riemann problem and additionally, under general initial conditions.
3. Trying models that explicitly include the effect of additional lane-changes that the TROW scheme will induce.
4. Extending the TROW scheme to more than 3 commodities, or to continuous VOT cases, and comparing the numerical results.

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