WAVE PROPAGATION SOLUTION FOR HIGHER ORDER MACROSCOPIC TRAFFIC MODELS IN THE PRESENCE OF INHOMOGENEITIES

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ABSTRACT

In general, the higher order macroscopic traffic model consists of partial differential equations (PDEs) that are of hyperbolic type. Accordingly, the standard well-posed theory of the hyperbolic system and corresponding numerical solutions are applied straightforwardly. In the presence of in-homogeneities such as on-and off-ramps, the higher order model has a form of PDEs with stiff source terms (i.e. high inflows and outflows at on-and off-ramps). It has been reported that attempt to use the standard solutions for such PDEs with stiff source terms may result in incorrect propagation speeds of discontinuities. To this end, this paper is aimed to introduce a novel method, namely, wave propagation, to simulate the higher order macroscopic traffic model in the presence of in-homogeneities. The proposed method is of Godunov-type finite volume method which adopts Riemann problems to determine the cell interface fluxes at each time step. In this paper, the source terms are augmented in the Riemann solutions to determine the propagation waves arising in the Riemann problems. The performance of the wave propagation method in traffic flow simulations is investigated in comparison with the existing methods via numerical examples. It is found that the proposed method is more accurate than the others under the same mesh size.

Keywords: higher order macroscopic model, Riemann solutions, wave propagation, stiff source terms.

INTRODUCTION

There are three types of modelling approach to describe the traffic operations: microscopic, mesoscopic and macroscopic. Microscopic models describe traffic flow at high level of detail such as the movement of individual vehicles, whereas macroscopic models represent traffic flow at low level of detail by aggregate traffic variables such as flow, mean speed and density. Mesoscopic models, on the other hand, deal with traffic flow through probabilistic terms. That is, traffic flow is described by the behaviour of a group of vehicles. This paper is dedicated to macroscopic models with main focus on the analysis of numerical solutions. In principle, macroscopic traffic models are used to replicate the average traffic behavior in terms of aggregate variables such as (time-space) flow and speed. There are two distinct methods to develop a macroscopic traffic model. The first order method is related to a so-called LWR-type models (Lighthill and Whitham; 1955, Richards, 1956) which are simple and allow one to represent a formation of shock wave but its assumption of a steady-state speed density relationship does not allow fluctuations around the equilibrium fundamental diagram. The steady-state speed density relationship is relaxed in the higher order models in which an additional dynamic equation for the mean speed or flow is proposed. Examples of higher order macroscopic models can be found in Hoogendoorn and Bovy (2001). Although the higher order models have been able to show some significant improvements over the first-order models in replicating the transitions between traffic congested states and non-linear phenomena (Schonhof and Helbing, 2007), it is difficult to implement a relevant numerical solution for such models. This paper focuses on the higher order macroscopic model which describes the dynamics of traffic flow on freeway in the presence of in-homogeneities in the following form:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S(U). \qquad (1)$$

In equation (1) $U = \begin{bmatrix} \rho \\ q \end{bmatrix}$, $F = \begin{bmatrix} q \\ E \end{bmatrix}$, $S = \begin{bmatrix} g^{\pm}(x,t) \\ G \end{bmatrix}$, where: $G = \frac{\rho\left(V^e - V\right)}{\tau}$. *$\rho=\rho(x,t)$*, *$q=q(x,t)$* and *$V=V(x,t)$* denote, respectively, the density, flow and mean speed at location *x* and time *t*. By definition *$q=\rho V$*. *$V^e= V^e(\rho,V)$* denotes the equilibrium speed function and τ denotes the relaxation time. $g^{\pm}(x,t)$ denotes the sources due to inflow (plus sign) and outflow (minus sign) at the on-and off-ramps, respectively. $E = \rho V^2 + P$, where *P* denotes the traffic pressure (Aw and Rascle, 2000), which reflects the anticipation of drivers to the downstream traffic situations. Different definitions of *P* and V^e lead to various higher order macroscopic models.

- Setting $V^e = V^e(\rho)$ and $P = \rho c_0^2$, where $c_0^2 = \frac{1}{2\rho\tau}\left|\frac{dV^e}{d\rho}\right|$, results in the model of Payne (1971) .
- Setting $V^e = V^e(\rho)$ and $P = \rho\Theta_0 - \mu\frac{\partial V}{\partial x}$, where μ denotes the so-called viscosity coefficient, results in the model of Kerner and Konhauser (1994).
- By setting $V^e = V^e(\rho,V)$ and $P = \rho\Theta$, where Θ is determined from the empirical relationship $\Theta = \alpha(\rho)V^2$, the model of Treiber et al. (1999) is obtained. Here, $\alpha(\rho)$ is a density-dependent function, estimated from empirical data.

It is straightforward to write equation (1) in the following conservation form:

$$\frac{\partial U}{\partial t} + J\frac{\partial U}{\partial x} = S(U), \qquad (2)$$

where $J = \begin{bmatrix} 0 & 1 \\ V^2 + P_\rho - \frac{VP_V}{\rho} & 2V + \frac{P_V}{\rho} \end{bmatrix}$ with $P_\rho = \frac{\partial P}{\partial \rho}, P_V = \frac{\partial P}{\partial V}$, denotes the Jacobian matrix having two eigenvalues (characteristic speeds):

$$\lambda^1 = V + \frac{P_V}{2\rho} + \sqrt{\left(\frac{P_V}{2\rho}\right)^2 + P_\rho}, \lambda^2 = V + \frac{P_V}{2\rho} - \sqrt{\left(\frac{P_V}{2\rho}\right)^2 + P_\rho} \qquad (3)$$

and two eigenvectors:

$$\Gamma^1 = \begin{bmatrix} 1 \\ \lambda^1 \end{bmatrix} \quad \Gamma^2 = \begin{bmatrix} 1 \\ \lambda^2 \end{bmatrix} \tag{4}$$

In general, $\lambda^2 < \lambda^1$ and these characteristic speeds are always real except the model of Kerner and Konhauser (1994) so the traffic equation (1) is of hyperbolic type. Consequently, the standard well-posed theory of the system of hyperbolic partial differential equations and the corresponding numerical solutions can be applied.

From equation (1) when the dissipative process (caused by the right hand side) is too fast compared to the conservative process (induced by the left hand side), the source term becomes a so-called stiff source term. This situation is fully observed in traffic flow due to the relatively high inflow from on-ramp or to the off-ramp. When the source term is stiff, the original traffic equation becomes an asymptotic reduced system (Chen et al., 1994). Numerical solution of traffic equation with stiff source term requires a careful numerical treatment of the source term so that the solution is well-balanced and robust on coarse mesh size (Leveque, 1998). This paper will concentrate on the treatment of the source term *S(U)* in the numerical solution and how this source term influences the accuracy of the numerical solution. The major contribution of this paper is to introduce a novel numerical solution, namely wave propagation, to the higher order traffic flow model in the presence of stiff source term (i.e. with high inflow from on-ramp or to off-ramp). The method is proposed in Leveque (1997) for hydrology and will be first time applied for traffic flow modelling with on- and off-ramps in this paper. In principle, the wave propagation method is of Godunov-type finite volume method which adopts Riemann problems to determine the cell interface fluxes at each time step. The source terms are then augmented in the Riemann solutions to determine the propagation waves arising in the Riemann problems.

STANDARD NUMERICAL SOLUTIONS

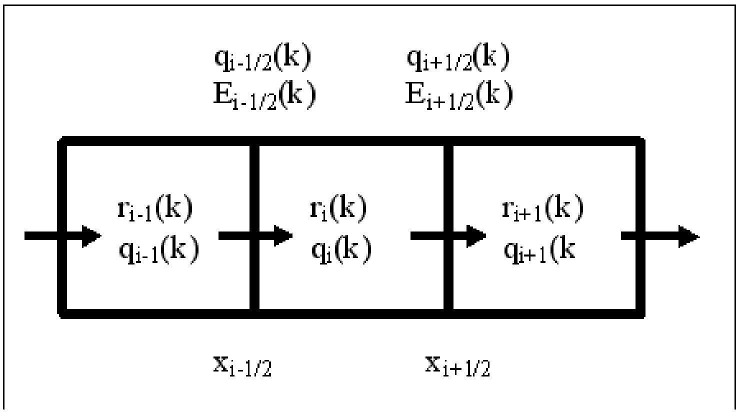

Figure 1 – Finite volume method for higher order macroscopic models

To simulate the traffic equation (1), the roadway is divided into many cells with equal length Δx (see Figure 1) whereas the simulation horizon is divided into time interval $t_k = k\Delta t$ with time step Δt . The Riemann problems are characterized by the following initial conditions:

$$U(x,t_k) = \begin{cases} U_i^k & \text{if } x < x_{i+1/2} \\ U_{i+1}^k & \text{if } x > x_{i+1/2} \end{cases}. \quad (5)$$

By definition, the cell average traffic variables are: $U_i^k = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x,t_k)dx$. By applying the approximation solver for the Riemann problems (5), we obtain:

$$\begin{aligned} &\text{Step 1: } \tilde{U}_i = U_i^k - \frac{\Delta t}{\Delta x}\left[F_{i+1/2}^k - F_{i-1/2}^k\right] \\ &\text{Step 2: } U_i^{k+1} = \tilde{U}_i + \Delta t S(\tilde{U}_i). \end{aligned} \quad (6)$$

In equation (6), $F_{i+1/2}^k$ denotes the numerical fluxes at cell interface $x_{i+1/2}$. Different numerical methods give rise to various functional forms of $F_{i+1/2}^k$ (see for more details). Step 1 represents an approximation solution for PDEs without source terms, where as step 2 is actually an approximation solution for ordinary partial differential equations.

The two steps method presented in equation (6) is adopted widely to simulate traffic flow models with source terms (Helbing and Treiber, 1999; Zhang, 2001; Jin and Zhang, 2003; Ngoduy et al., 2004a, 2004b; Zhang et al., 2008). However, it has been reported in Levecque and Yee (1990), Pember (1993), Chalabi and Qiu (2000 that the two steps method has a problem when the flux gradients and the source terms are cancelled out each other (i.e. $\frac{\partial F}{\partial x} = S$) . It has been observed that this problem may result in incorrect propagation speeds of discontinuities, which may then cause erroneous traffic flow predictions, for example traffic jams that never occur in reality. To this end, the rest of this paper is devoted to the implementation of a distinct method, namely wave propagation, to accurately simulate traffic equation (1) in the presence of high inflow and outflow at on-and off-ramps.

WAVE PROPAGATION SOLUTION

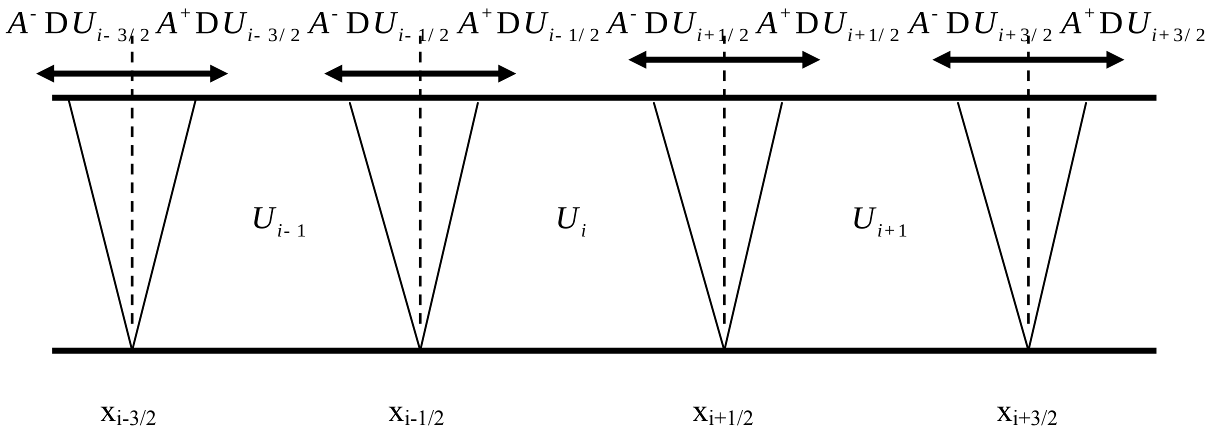

Figure 2- Structure of the wave propagation solution.

The wave propagation method is based on solving Riemann problems (5) for the wave structure. Then the flux splitting technique that generalizes the notation of flux difference splitting from conservation law is introduced. Figure 2 illustrates the left-going and right-going flux splitting at cell interfaces that capture the net effect of all left- and right-going waves.

Let $W_{i-1/2}^{j}$ denote the j^{th} waves ($j \in J$) at cell interface $x_{i-1/2}$, which occur due to the jump in the traffic variables. Let us assume that $U_{i+1}^{k} - U_{i}^{k}$ can be decomposed as:

$$U_{i+1}^{k} - U_{i}^{k} = \sum_{j=1}^{J} W_{i+1/2}^{j} = \sum_{j=1}^{J} \gamma_{i+1/2}^{j} \Gamma_{i+1/2}^{j}, \tag{7}$$

where J denotes the number of waves (in this paper $J=2$) each of which is associated with a wave speed λ^{j}, which is determined in equation (3). $\gamma_{i+1/2}^{j}$ is a coefficient, determined as:

$$\begin{aligned} \gamma_{i+1/2}^{1} &= -\frac{\lambda_{i+1/2}^{2}}{\lambda_{i+1/2}^{1} - \lambda_{i+1/2}^{2}}\left(\rho_{i+1} - \rho_{i}\right) + \frac{1}{\lambda_{i+1/2}^{1} - \lambda_{i+1/2}^{2}}\left(q_{i+1} - q_{i}\right), \\ \gamma_{i+1/2}^{2} &= \frac{\lambda_{i+1/2}^{1}}{\lambda_{i+1/2}^{1} - \lambda_{i+1/2}^{2}}\left(\rho_{i+1} - \rho_{i}\right) - \frac{1}{\lambda_{i+1/2}^{1} - \lambda_{i+1/2}^{2}}\left(q_{i+1} - q_{i}\right). \end{aligned} \tag{8}$$

In equation (7) $\Gamma_{i+1/2}^{j}$ denotes the eigenvector determined by $\lambda_{i+1/2}^{j}$. In equation (8), $\lambda_{i+1/2}^{j} = \lambda^{j}\left(\rho_{i}^{k}, \rho_{i+1}^{k}, V_{i+1/2}^{k}\right)$ where $V_{i+1/2}^{k}$ is determined by Roe approximation:

$$V_{i+1/2}^{k} = \frac{V_{i}^{k}\sqrt{\rho_{i}^{k}} + V_{i+1}^{k}\sqrt{\rho_{i+1}^{k}}}{\sqrt{\rho_{i}^{k}} + \sqrt{\rho_{i+1}^{k}}}. \tag{9}$$

Let $Z_{i+1/2}^{j}$ denote the j^{th} waves ($j=1,2$) at cell interface $x_{i+1/2}$, which occur due to the *jump in the fluxes* including the net contribution of source terms. The jump in these fluxes can be decomposed as:

$$F_{i+1}^{k} - F_{i}^{k} - \Delta x S_{i}^{k} = \sum_{j=1}^{J} Z_{i+1/2}^{j} = \sum_{j=1}^{J} \beta_{i+1/2}^{j} \Gamma_{i+1/2}^{j}, \tag{10}$$

By using the Rankine-Hugoniot condition across each wave at the cell interface $x_{i+1/2}$ ($F_{i+1}^{k} - F_{i}^{k} - \Delta x S_{i}^{k} = \sum_{j=1}^{J} \lambda_{i+1/2}^{j} W_{i+1/2}^{j}$) we can compute the wave coefficient $\beta_{i+1/2}^{j}$ as:

$$\begin{aligned} \beta_{i+1/2}^{1} &= -\frac{\lambda_{i+1/2}^{2}}{\lambda_{i+1/2}^{1} - \lambda_{i+1/2}^{2}}\left(q_{i+1} - q_{i} - \Delta x g_{i}\right) + \frac{1}{\lambda_{i+1/2}^{1} - \lambda_{i+1/2}^{2}}\left(E_{i+1} - E_{i} - \Delta x G_{i}\right), \\ \beta_{i+1/2}^{2} &= \frac{\lambda_{i+1/2}^{1}}{\lambda_{i+1/2}^{1} - \lambda_{i+1/2}^{2}}\left(q_{i+1} - q_{i} - \Delta x g_{i}\right) - \frac{1}{\lambda_{i+1/2}^{1} - \lambda_{i+1/2}^{2}}\left(E_{i+1} - E_{i} - \Delta x G_{i}\right). \end{aligned} \tag{11}$$

It can be seen from equation (11) that the source terms g and G are now embedded in the wave solution at cell interface through the calculation of the wave coefficient $\beta_{i+1/2}^{j}$.

Now let $U_{i+1/2}^{k}$ denote the value of the Riemann solution along the interface $x_{i+1/2}$, then Godunov's method can be written as:

$$U_{i}^{k+1} = U_{i}^{k} - \frac{\Delta t}{\Delta x}\left[f\left(U_{i+1/2}^{k}\right) - f\left(U_{i-1/2}^{k}\right)\right] \tag{12}$$

Let $A^{+}\Delta U_{i-1/2}^{k}$ denote the net contribution of the right moving waves to the traffic evolution of cell i from the left interface $x_{i-1/2}$, whereas $A^{-}\Delta U_{i+1/2}^{k}$ present the net contribution of the left moving waves to the traffic evolution of cell i from the right interface $x_{i+1/2}$ (see Figure 2). Let us define:

$$A^{+}\Delta U^{k}_{i-1/2} = f\left(U^{k}_{i}\right) - f\left(U^{k}_{i-1/2}\right), A^{-}\Delta U^{k}_{i+1/2} = f\left(U^{k}_{i+1/2}\right) - f\left(U^{k}_{i}\right), \tag{13}$$

equation (12) can be rewritten in the following form:

$$U^{k+1}_{i} = U^{k}_{i} - \frac{\Delta t}{\Delta x}\left[A^{+}\Delta U^{k}_{i-1/2} + A^{-}\Delta U^{k}_{i+1/2}\right] \tag{14}$$

In equation (14), the source terms are embedded in the Riemann solutions through equation (10) and equation (11) when determining the right and left moving wave contribution $A^{+}\Delta U^{k}_{i-1/2}$ and $A^{-}\Delta U^{k}_{i+1/2}$, respectively. The right and left moving wave contribution are determined as:

$$A^{+}\Delta U^{k}_{i-1/2} = \sum_{j=1}^{J}\left(Z^{j}_{i-1/2}\right)^{+}, A^{-}\Delta U^{k}_{i+1/2} = \sum_{j=1}^{J}\left(Z^{j}_{i+1/2}\right)^{-}. \tag{15}$$

It is worth noticing from equation (3) that the first characteristic speed $\lambda^{1} > 0$, so equation (15) is determined only from the sign of the second characteristic speed λ^{2}:

$$A^{+}\Delta U^{k}_{i-1/2} = \begin{cases} \beta^{1}_{i-1/2}\Gamma^{1}_{i-1/2} + \beta^{2}_{i-1/2}\Gamma^{2}_{i-1/2} & \text{if } \lambda^{2}_{i-1/2} > 0 \\ \beta^{1}_{i-1/2}\Gamma^{1}_{i-1/2} & \text{if } \lambda^{2}_{i-1/2} \le 0 \end{cases},$$

$$A^{-}\Delta U^{k}_{i+1/2} = \begin{cases} \beta^{2}_{i+1/2}\Gamma^{2}_{i+1/2} & \text{if } \lambda^{2}_{i+1/2} < 0 \\ 0 & \text{if } \lambda^{2}_{i+1/2} \ge 0 \end{cases}. \tag{16}$$

Equations (14) and (16) complete the wave propagation solution for the traffic flow model (1) in the presence of on- and off-ramps. Numerical tests will be carried out in the next section to investigate the performance of the proposed numerical solution against the standard two steps solution.

NUMERICAL STUDY

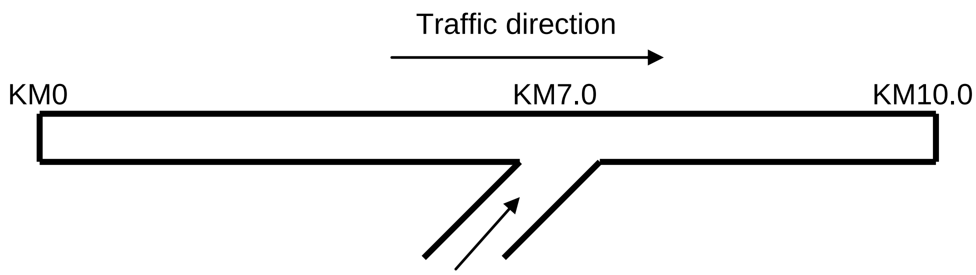

Figure 3- Layout of a freeway with an on-ramp

In this section we investigate the performance of the wave propagation method for simulation of a 10km-freeway with an on-ramp which is located at KM7.0 (Figure 3). The model of Payne (1971) is chosen for the study. The open boundary condition is adopted for our simulation with the entry flow=6*00 veh/h*, entry speed=10*0 km/h*, ramp flow=*1200 veh/h*. To test the numerical convergence of the wave propagation solution, we use different mesh sizes for the same boundary and initial conditions, such as *[25m,0.5sec], [50m,1sec], [100m,2sec], [200m,4sec]*. Let us assume that the smallest mesh size represents the convergent solution, which is obtained in the limit $\lim(\Delta x \to 0, \Delta t \to 0)$. The so-called *Euclidian* root mean square error (ERMSE) norm is used to illustrate the accuracy of the solution:

$$\|e^{\rho}(k)\| = \sqrt{\frac{1}{I}\sum_{i=1}^{I}\left[\rho_i^{sim}(k) - \rho_i^{exact}(k)\right]^2}, \quad \|e^{V}(k)\| = \sqrt{\frac{1}{I}\sum_{i=1}^{I}\left[V_i^{sim}(k) - V_i^{exact}(k)\right]^2}, \qquad (17)$$

where $\rho_i^{sim}(k), V_i^{sim}(k)$ are density and speed obtained at cell *I* and time instant *k* from different mesh sizes. $\rho_i^{exact}(k), V_i^{exact}(k)$ are density and speed obtained at cell *I* and time instant *k* from the smallest mesh size. *I* denotes the total number of cells.

Simulation results of the proposed numerical solution are described in Figure 4 and Figure 5. Figure 4 illustrates the density and speed profiles reproduced from different mesh sizes. It is obvious from Figure 4 that the large mesh size, the large smoothing effects with respect to the upfront jam. However, the ERMSE is bounded and converges for different mesh sizes (Figure 5). In order to study the accuracy of the proposed solution, we have also simulated the same freeway with the same model, inputs and model parameters using the Harten, van Leer, Lax and Einfeldt (HLLE) two steps scheme proposed in Ngoduy et al. (2004a) for traffic equations. The ERMSE produced by HLLE method is shown in Figure 6. It is clear from Figure 5 and Figure 6 as well as Table 1 that the proposed numerical method is more accurate than the two steps HLLE scheme in our example. Further experiments with other two steps numerical method are needed to support the findings.

Table 1- Euclidian root mean square errors for different mesh sizes

	Mesh size	HLLE method				Wave propagation method			
		t=300	t=600	t=900	t=1200	t=300	t=600	t=900	t=1200
Density (veh/km)	50m, 1sec	3.2	3.6	3.7	3.9	2.2	2.2	2.2	2.5
	100m, 2sec	6.6	7.3	7.7	7.9	4.5	4.4	4.5	5.2
	200m, 4sec	10.5	11.9	12.4	13.0	8.8	7.7	7.4	8.5
Speed (km/h)	50m, 1sec	5.2	4.8	4.8	5.0	2.5	2.2	2.7	3.3
	100m, 2sec	10.0	9.8	9.9	10.2	5.0	3.3	5.4	6.4
	200m, 4sec	15.7	15.7	15.7	16.2	10.6	6.4	7.9	9.4

CONCLUDING REMARKS

In general, the higher order macroscopic traffic flow model for freeways with on-and off-ramp is a system of (hyperbolic) partial differential equations with stiff source terms when the inflow/outflow at on-and off-ramp is relatively high. Consequently, the dissipative process caused by the right hand side is too fast compared to the conservative process caused by the left hand side. The problem with stiff source terms requires a careful numerical treatment of the source term so that the solution is well-balanced and robust on coarse mesh size. This paper has presented a robust numerical solution, namely wave propagation, for a class of higher macroscopic traffic flow models in the presence of on-and off-ramp. The wave propagation solution is based on solving the Riemann problems at cell interfaces where the source terms are augmented into the jump of the fluxes using the Rankine-Hugoniot condition across each wave. Therefore, the proposed solution is a one step numerical approximation of the traffic equations with source terms, which overcome the drawback of the two step method when the source terms are stiff as reported in literature of computational physics. Numerical tests have indicated that the wave propagation method is robust with coarse mesh sizes and the numerical convergence is proven by through the computation of the Euclidian root mean square errors. It has also been shown that the proposed numerical solution is more accurate than a standard two steps solution given the same mesh sizes in our example. However, more extensive experiments with other two steps numerical method are needed to support the findings.

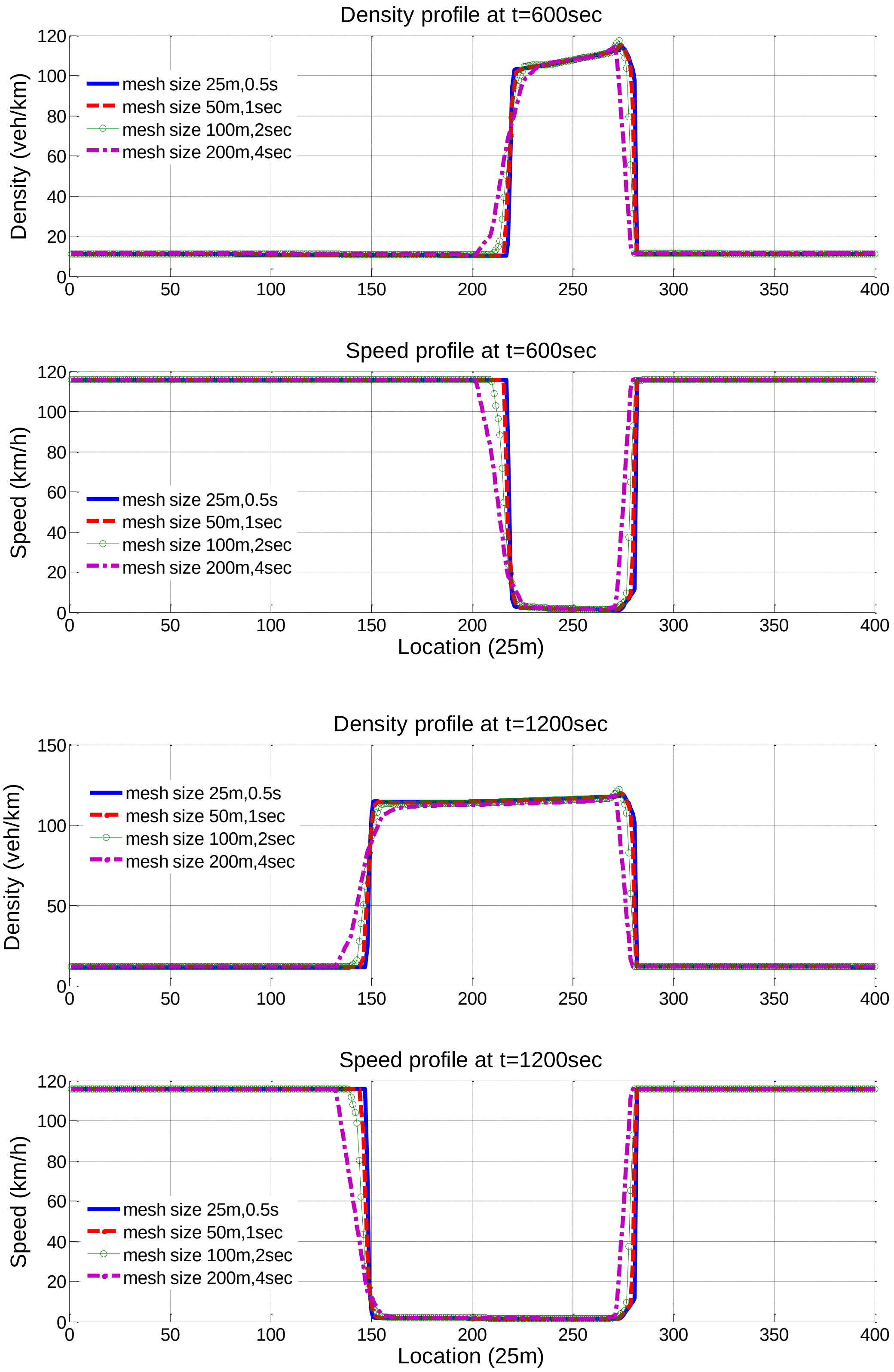

Figure 4- Convergence test with different mesh sizes for density and speed by wave propagation method

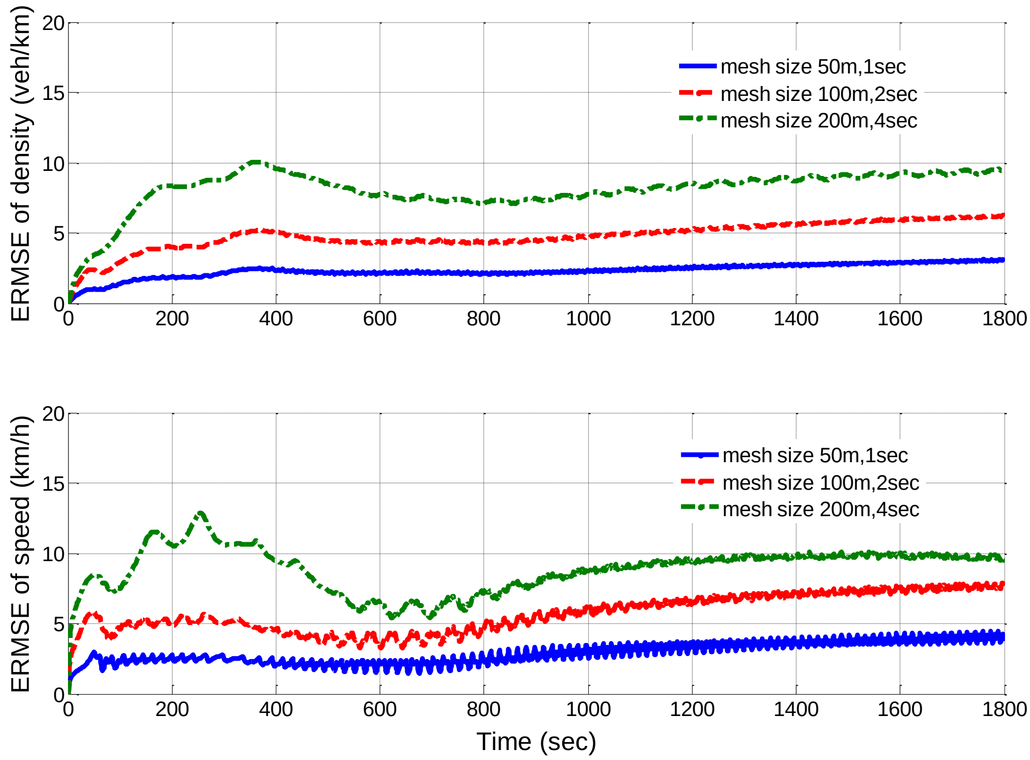

Figure 5- Euclidian root mean square errors of density and speed by wave propagation method.

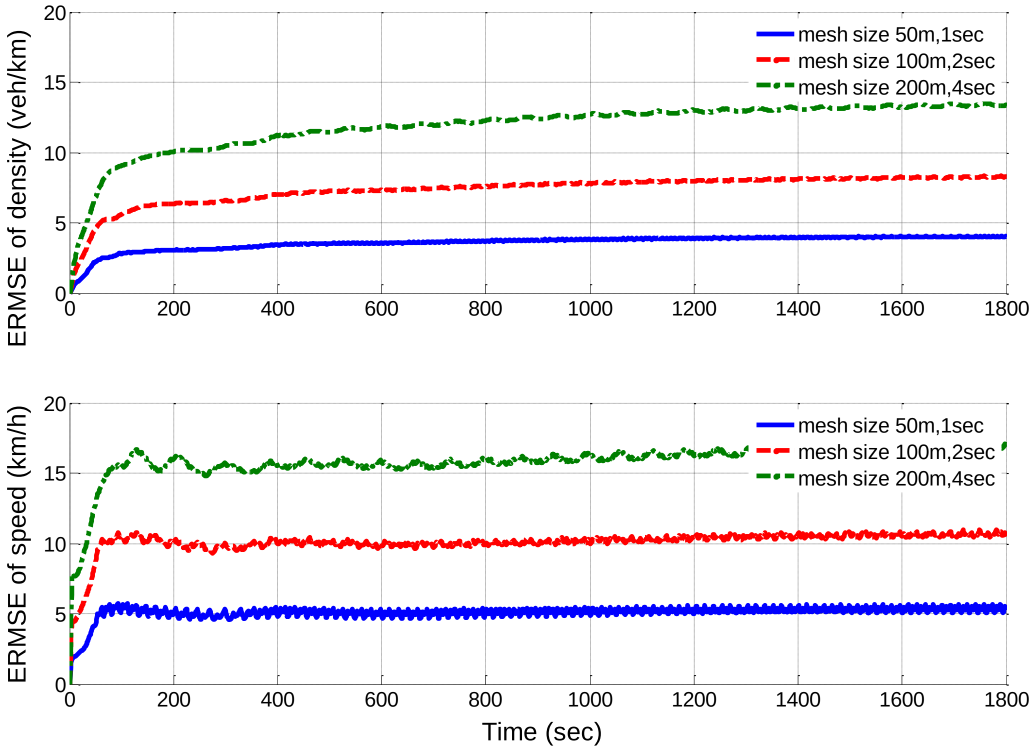

Figure 6- Euclidian root mean square errors of density and speed by two step HLLE method.

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