TRIAL-AND-ERROR METHOD FOR CORDON-BASED CONGESTION PRICING SCHEME WITH PROBIT-BASED STOCHASTIC USER EQUILIBRIUM CONSTRAINTS

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ABSTRACT

An effective toll pattern for the cordon-based congestion pricing scheme aims to levy a toll on each entry of a cordon such that the number of vehicle passing through the entries during a peak hour does not exceed a predetermined entry-specific threshold (a threshold constraint), and free passage on the entry is granted if the hourly traffic volume is strictly less than the threshold. To determine the effective toll pattern with the deterministic user equilibrium constraints, an engineering-oriented trial-and-error method only using entry-specific traffic counts has been recently put forward. Thus, this paper investigates availability of this trial-and-error method for the case that behavior of drivers in route choice obeys the probit-based stochastic user equilibrium (SUE) principle. After building a minimization model for the elastic demand probit-based SUE problem with the threshold constraints, this paper first shows that product of value of time (VOT) and optimal Lagrangian multipliers with respect to the threshold constraints is an effective toll pattern. This paper thus proceeds to rigorously demonstrate global convergence of the trial-and-error in estimating the effective toll pattern with the probit-based SUE constraints. Such availability is finally evaluated by a numerical example.

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1. INTRODUCTION

The first- and second- best pricing principles are well recognized/accepted by researchers for determining an appropriate toll pattern to mitigate traffic congestion (Small, 1992; Lewis, 1993; Yang and Huang 2005; Lawphongpanich et al. 2006). The first-best principle aims to maximize total social benefit by levying a toll on each road (link) of a transportation network. It is thus unrealistic because each link is required to be charged. The second-best principle imposes a toll on each link in a subset of all the links such that a system performance index is minimized. The first-best pricing solution can be obtained by solving a deterministic user equilibrium (DUE) or stochastic user equilibrium (SUE) problem (Huang and Yang 1996; Yang and Huang, 1998; Hearn et al. 1998; Hearn et al. 2002; Yin and Yang, 2004; Maher et al., 2005). While, the second-best pricing solution can be estimated by solving a bi-level programming model in which the lower level problem is a DUE or SUE formulation (Verhoef, et al. 1996; Verhoef, 2002; Zhang, 2003; Chen and Bernstein, 2004; Zhang, 2003; Sumalee, 2004).

As a generalized second-best pricing principle, the cordon-based congestion pricing scheme has been not only examined by researchers (e.g. Zhang and Yang 2004; Sumalee, et al. 2005; Sumalee, 2007), but also adopted by the urban congestion pricing practitioners. For example, the electronic road pricing system (ERP) of Singapore and the congestion pricing scheme in London. The cordon-based congestion pricing scheme includes a charging cordon comprising a group of entries to a designated congestion area such as the central business district (CBD). Vehicles traversing any of these entries are charged during the peak hours. It, in reality, reduces traffic congestion within a designated urban area by limiting the number of vehicles entering the area. This goal can be realized by an entry-specific toll pattern fulfilling two conditions as follows: (a) the number of vehicles passing through an entry during a peak hour does not exceed a given threshold associated with the entry; (b) free passage is granted if the hourly traffic volume is strictly less than the threshold. Condition (a) reflects effectiveness of the cordon-based congestion pricing scheme, and condition (b) ensures the equity to some extent because those vehicles do not contribute the external congestion of the designated urban area. For the sake of presentation, an entry-specific toll pattern satisfying the above two conditions is referred to as an effective toll pattern. Note that these two conditions make the effective toll pattern significantly different from the classical second-best pricing solution. It is thus important to seek for solution algorithms that can efficiently find an effective toll pattern in view of the reality of cordon-based congestion pricing scheme.

Regarding the effective toll pattern with the DUE constraints, Ferrari (1995 & 1997), Yang and Lam (1996) and Larsson and Patriksson (1999) have proposed different algorithms to determine an effective toll pattern. These algorithms, however, necessitate explicit and precise mathematical expressions of link travel time functions, origin-destination (OD) demand functions and drivers' value of time (VOT). Yet, none of these functions and VOT could be acquired easily and accurately in practice. It is a challenge to find analytical expressions of OD demand functions and precisely estimate the VOT because they both reflect psychological behavior of human beings. Despite of this, traffic counts on each entry during the peak hour for the cordon-based congestion pricing system can be automatically and precisely acquired by the electronic toll collection devices installed on the entry. For example, the ERP in Singapore uses a dedicated short-range radio communication system to deduct ERP charges from smart-cards inserted in the in-vehicle units each time they pass though a ERP gantry (an entry).

Without resorting of link travel time functions, OD demand functions and VOT, Meng et al. (2005) developed a convergent trial-and-error method to estimate an effective toll pattern with the DUE constraints. The trial-and-error method works iteratively as follows: impose an entry-specific toll pattern at each trial and then count the number of vehicles passing through each entry during the peak hour; subsequently adjust the entry-specific toll pattern according to the difference between the hourly traffic counts on each entry and a threshold predetermined for the entry. Since this trial-and-error method only needs the entry-specific traffic counts, it is more implementable in practice. In reality, this trial-and-error method enriches the emerging exploration of the trial-and-error methods estimating the conventional first- and second- best pricing solutions (Downs, 1993; Vickrey, 1993; Li, 2002; Yang et al, 2004 and 2005; Xu, 2006; Zhao and Kockelman, 2006; Han and Yang, 2008). It is well-known that the probit-based SUE conditions play the same role as the DUE conditions in characterizing the route choice behaviour of drivers (Sheffi, 1995; Patriksson, 1994). It is thus interesting to examine whether or not the trial-and-error method proposed by Meng et al. (2005) is available for estimating an effective toll pattern with the probit-based SUE constraints.

In this paper, a minimization model is first built for probit-based SUE problem with the entry-specific threshold constraints and elastic demand, and properties of its solution are investigated. Second, we show that product of VOT and optimal Lagrangian multiplier associated with the entry-specific threshold constraints yield an effective toll pattern. Third, a Lagrangian dual formulation of the proposed minimization model is established. Finally, we demonstrate that trial-and-error method proposed by Meng et al. (2005) is still available for solving the effective toll pattern in this case.

The remainder of this paper is organized as follows. Section 2 mathematically defines the effective toll pattern with the probit-based SUE constraints. Section 3 gives a minimization model for the elastic demand probit-based SUE problem with the entry-specific threshold constraints and analyzes its solution properties. Section 4 shows availability of the trial-and-error method in estimating an effective toll pattern with the Probit-based SUE constraints. Section 5 numerically evaluates the availability of the trial-and-error method. Conclusions are presented in Section 6.

2. THE EFFECTIVE TOLL SOLUTION WITH PROBIT-BASED SUE CONSTRAINTS

To properly express the effective toll solution with the probit-based SUE constraint, notations and the conventional assumptions are first presented below. A rigorous mathematical representation for the effective toll solution is then given.

2.1 Notations

N : Set of nodes for an urban transportation network.
A: Set of links for the network.
$\bar{A}$: Set of the entries (links) to a traffic attractive area, such as CBD, secured by a cordon-based congestion pricing scheme, and $\bar{A} \subseteq A$.
W : Set of the origin-destination (OD) pairs.
$\bar{q}_w$: Total value of the traffic demand between OD pair $w \in W$.
R_w: Set of all the paths between OD pair $w \in W$.
δ_{ak}^w : $\delta_{ak}^w = 1$ if path $k \in R_w$ between OD pair $w \in W$ traverses link $a \in A$, $\delta_{ak}^w = 0$ otherwise.
f_{wk} : Traffic flow on path $k \in R_w$ between OD pair $w \in W$.
$\mathbf{f}_w$: Row vector of traffic flows of all the paths between OD pair $w \in W$, namely, $\mathbf{f}_w = \left(f_{wk}, k \in R_w \right)$
$\mathbf{f}$: Row vector of traffic flows of all the paths in the network, i.e., $\mathbf{f} = \left(\mathbf{f}_w, w \in W\right)$.
v_a : Traffic flow on link $a \in A$.
τ_a : Toll imposed on entry $a \in \bar{A}$.
$\boldsymbol{\tau}$; Row vector of all the entry-specific tolls, that is, $\boldsymbol{\tau} = \left(\tau_a, a \in \bar{A}\right)$.
H_a : Physical capacity of traffic flow on the entry $a \in \bar{A}$.
$\bar{H}_a$: Predetermined threshold of traffic flow on the entry $a \in \bar{A}$.

2.2 Assumptions

A1. Travel time on each link $a \in A$ is a non-negative, non-decreasing and continuously differentiable function of its traffic flow v_a , denoted by $t_a\left(v_a\right)$.

A2. Travel time on each link $a \in A$ perceived by the drivers, denoted by $\tilde{T}_a\left(v_a\right)$, follows a normal distribution $N\left(\tilde{t}_a\left(v_a\right), \theta t_a^0\right)$, in which t_a^0 is free-flow travel time on the link, θ is a proportionality parameter, and $\tilde{t}_a\left(v_a\right)$ is the actual generalized link travel time function defined by

$$\tilde{t}\left(v_a\right) = \begin{cases} t_a\left(v_a\right) + \left(\tau_a / \beta\right), & a \in \bar{A} \\ t_a\left(v_a\right), & a \in A \setminus \bar{A} \end{cases}, \quad a \in A \tag{1}$$

where parameter β is the value of time (VOT) of drivers.

A3. Driver's behaviour in route choice obeys the probit-based SUE principle in terms of the users' perceived travelling utility on each path (Daganzo, 1982):

$$\begin{aligned} \tilde{U}_{wk}\left(\mathbf{f}\right) &= \bar{U}_w - \tilde{C}_{wk}\left(\mathbf{f}\right), k \in R_w, w \in W \\ \tilde{C}_{wk}\left(\mathbf{f}\right) &= \sum_{a \in A} \tilde{T}_a\left(v_a\right) \delta_{ak}^w \end{aligned} \tag{2}$$

where $\bar{U}_w$ is the initial utility value representing the benefit that users can gain from their trip, which is constant and can be treated as the mean utility of travelling across all the users on each OD pair. $\tilde{C}_{wk}(\mathbf{f})$ is the users' perceived path travel time .

A4. Travel demand between each OD pair is elastic, namely, each user makes their travelling decision based on their perceived utility on each path $\tilde{U}_{wk}$, and if this value is negative on all the paths, they will give up their journey. We further assume that each OD pair is connected by a dummy link, and all the excess flows would be loaded to this link. In this paper, these excess flows are denoted by $e_w, w \in W$. And the total demand on the real network and the dummy link is $\bar{q}_w$, i.e., $e_w + \sum_{k \in R_w} f_{wk} = \bar{q}, w \in W$. Note that this assumption was originally made by Bellei et al. (2002) and Connors et al. (2007).

In eqn. (1) the monetary entry-specific toll is converted into the travel time using VOT (Yang and Huang, 2005), which is assumed to be consistent for all the users. Regarding the probit-based SUE principle made in Assumptions 2 and 3, the perceived generalized path travel time is equal to the sum of the perceived link travel times, shown in eqn. (2), which is proposed by Sheffi (1985) as an alternative representation of the conventional perceived path travel time. In reality, eqn. (2) can be rewritten as:

$$\tilde{U}_{wk}(\mathbf{f}) = \bar{U}_w - \tilde{c}_{wk}(\mathbf{f}) - \xi_{wk}, k \in R_w, w \in W \tag{3}$$

where $\tilde{c}_{wk}(\mathbf{f})$, the actual generalized travel time on path $k \in R_w$, is defined by

$$\tilde{c}_{wk}(\mathbf{f}) = c_{wk}(\mathbf{f}) + \sum_{a \in \bar{A}} (\tau_a / \beta) \delta_{ak}^w \tag{4}$$

where $c_{wk}(\mathbf{f})$, the actual path travel time, is equal to the sum of actual travel times of links on the path, i.e.,

$$c_{wk}(\mathbf{f}) = \sum_{a \in A} t_a(v_a) \delta_{ak}^w \tag{5}$$

and ξ_{wk} is a normal distributed random error term with mean of zero and variance:

$$Var(\xi_{wk}) = \theta \sum_{a \in A} t_a^0 \delta_{ak}^w \tag{6}$$

More importantly, the covariance between any two of these random errors can be expressed by

$$\mathrm{cov}(\xi_{wk,} \xi_{wr}) = \theta \sum_{a \in A} t_a^0 \delta_{ak}^w \delta_{ar}^w, \forall r, k \in R_w, w \in W \tag{7}$$

According to eqn. (2), it can be seen that $\sum_{a \in A} t_a^0 \delta_{ak}^w \delta_{ar}^w$ in the right hand side of eqn. (7) is the sum of the free-flow travel time of overlapped links between two paths r and k .

In light of assumption 4, the travel costs on the dummy links are zero and have zero variance, and they are taken into the probit-based SUE loading. So we have transformed the elastic demand problem into a fixed demand one by this network representation.

2.3 Mathematical Representation of the Effective Toll Solution

As described previously for the cordon-based congestion pricing scheme implemented in Singapore, the effective toll solution is to restrain traffic flow on each entry to a predetermined threshold by levying an entry-specific toll pattern. If traffic flow on an entry is lower than the threshold, the effective toll solution should give free travel right to those drivers traversing this entry, because they have not made more external congestion to the secured area in view of the threshold requirement. With the assumption of probit-based SUE principle for driver's behavior in the route choice, the effective toll solution denoted by a row vector $\boldsymbol{\tau}^* = \left(\tau_a^* : a \in \bar{A}\right)$ can be mathematically expressed as follows:

$$v_a\left(\boldsymbol{\tau}^*\right) \le \bar{H}_a, \quad a \in \bar{A} \tag{8}$$

$$\tau_a^* \times \left(v_a\left(\boldsymbol{\tau}^*\right) - \bar{H}_a \right) = 0, \quad a \in \bar{A} \tag{9}$$

$$\tau_a^* \ge 0, a \in \bar{A} \tag{10}$$

where $v_a\left(\boldsymbol{\tau}^*\right)$ is the probit-based SUE traffic flow on entry $a \in \bar{A}$ as a response result to the effective toll solution.

Eqn. (8) expresses the traffic control goal of the effective toll solution, and eqn. (9) describes the free travel right for those drivers passing through the entry when the traffic flow on an entry is less than $\bar{H}_a$. As any entry-specific toll should not be a negative number, eqn. (10) comes out.

3. MATHEMATICAL MODELS

According to the existing road pricing studies, such as Ferrari (1995 & 1997), Yang and Lam (1996) and Meng et al. (2005), it can be seen that optimal Lagrangian multipliers (with respect to the capacity constraints of the minimization model developed for the DUE traffic assignment) follow the form of Effective Toll Pattern, eqns. (8)-(10). It is interesting to explore if such a relation is still available for the probit-based SUE traffic assignment model with capacity constraints. To do so, we first need to seek a minimization model for the probit-based SUE traffic assignment with link capacity constraints.

More recently, Meng et al. (2008) have developed the following minimization model for the general SUE traffic assignment problem with capacity constraints and fixed-demand:

$$\min z_1\left(\mathbf{f}\right) = \sum_{w \in W} q_w S_w\left(\bar{\mathbf{c}}_w\left(\mathbf{f}\right) + \bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)\right) - \sum_{w \in W} \sum_{k \in R_w} \bar{d}_{wk}\left(\mathbf{f}_\mathbf{w}\right) f_{wk} \tag{11}$$

subject to

$$v_a \le H_a, a \in A \tag{12}$$

$$v_a = \sum_{w \in W} \sum_{k \in R_w} f_{wk} \delta_{ak}^w, a \in A \tag{13}$$

$$\sum_{k \in R_w} f_{wk} = q_w, w \in W \tag{14}$$

$$f_{wk} \ge 0, k \in R_w, w \in W \tag{15}$$

where H_a is the physical capacity of link $a \in A$, and the two row vectors $\bar{\mathbf{c}}_w\left(\mathbf{f}\right)$ and $\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)$ associated with OD pair $w \in W$ are elaborated as follows:

A hypothetical link travel time function, denoted by $\bar{t}_a\left(v_a\right), a \in A$, is first defined on each link:

$$\bar{t}\left(v_a\right)=\begin{cases}\dfrac{\int_0^{v_a} t_a\left(x\right)dx}{v_a}, v_a>0 \\ t_a\left(0\right), \qquad v_a=0\end{cases} \tag{16}$$

And $\bar{\mathbf{c}}_w\left(\mathbf{f}\right)=\left(\bar{c}_{wk}\left(\mathbf{f}\right),k\in R_w\right)$ is the row vector of the hypothetical path travel time functions of all paths between OD pair $w\in W$, in which this hypothetical path travel time function for path $k\in R_w$ follows this expression:

$$\bar{c}_{wk}\left(\mathbf{f}\right)=\sum_{a\in A}\bar{t}_a\left(v_a\right)\delta_{ak}^{w} \tag{17}$$

As to $\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)$, it is another row vector, i.e., $\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)=\left(\bar{d}_{wk}\left(\mathbf{f}_w\right):k\in R_w\right)$. And $\bar{d}_{wk}\left(\mathbf{f}_w\right)$ is a continuously differentiable function of the traffic flows on all the paths between the OD pair w, so that for any feasible path flow pattern $\mathbf{f}=\left(f_{wk},w\in W,k\in R_w\right)$, the following conditions can be satisfied (Maher, 2005).

$$p_{wk}\left(\bar{\mathbf{c}}_w\left(\mathbf{f}\right)+\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)\right)=\frac{f_{wk}}{q_w},k\in R_w \tag{18}$$

where, $p_{wk}\left(\bar{\mathbf{c}}_w\left(\mathbf{f}\right)+\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)\right)$ is the probability that the path k is perceived as the shortest one by drivers, among all the paths connecting OD pair w, with respect to the assumed path travel time pattern, $\bar{\mathbf{c}}_w\left(\mathbf{f}\right)+\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)$, namely,

$$p_{wk}\left(\bar{c}_w\left(\mathbf{f}\right)+\bar{d}_w\left(\mathbf{f}_w\right)\right)=\Pr\left(\bar{c}_{wk}\left(\mathbf{f}\right)+\bar{d}_{wk}\left(\mathbf{f}_w\right)+\xi_{wk}\le\bar{c}_{wr}\left(\mathbf{f}\right)+\bar{d}_{wr}\left(\mathbf{f}_w\right)+\xi_{rk},\forall r\in R_w,r\neq k\right) \tag{19}$$

$S_w\left(\bar{\mathbf{c}}_w\left(\mathbf{f}\right)+\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)\right)$, known as satisfaction function with respect to the assumed path travel time pattern $\bar{\mathbf{c}}_w\left(\mathbf{f}\right)+\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)$, is the mean value of the perceived minimal path travel time, i.e.,

$$S_w\left(\bar{\mathbf{c}}_w\left(\mathbf{f}\right)+\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)\right)=E\left[\min_{k\in R_w}\left(\bar{c}_{wk}\left(\mathbf{f}\right)+\bar{d}_{wk}\left(\mathbf{f}_w\right)+\xi_{wk}\right)\right] \tag{20}$$

The satisfaction function possesses the following important property (see, Section 12.1 of Sheffi (1985)):

$$\frac{\partial S_w\left(\bar{\mathbf{c}}_w\left(\mathbf{f}\right)+\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right)\right)}{\partial\left(\bar{c}_{wk}\left(\mathbf{f}\right)+\bar{d}_{wk}\left(\mathbf{f}_w\right)\right)}=p_{wk}\left(\bar{c}_w\left(\mathbf{f}\right)+\bar{d}_w\left(\mathbf{f}_w\right)\right),k\in R_w \tag{21}$$

3.1 Minimization Model for Probit-based SUE Problem with Threshold Constraints and Elastic Demand

As per Assumption 4, the probit-based SUE problem with elastic demand can be converted into the fixed demand case, by adding a dummy link between each OD pair. And the excess demand (users that drop their trip plan) would be loaded to this dummy link. Thus, based on the augmented network, satisfaction function in terms of utility rather than travel time is defined as:

$$S_w\left(U_w-\bar{\mathbf{c}}_w\left(\mathbf{f}\right)-\bar{\mathbf{d}}_w\left(\mathbf{f}_w\right),0-\bar{d}_w\left(e_w\right)\right)=E\left[\max_{k\in R_w}\left(U_w-\bar{c}_{wk}\left(\mathbf{f}\right)-\bar{d}_{wk}\left(\mathbf{f}_w\right)-\xi_{wk},0-\bar{d}_w\left(e_w\right)\right)\right] \tag{22}$$

where $\bar{d}_w\left(e_w\right)$ is defined on the dummy link between any OD pair $w\in W$, such that for any flow pattern $\left(\mathbf{f},\mathbf{e}\right)=\left(f_{wk},k\in R_w,,w\in W;e_w,w\in W\right)$ in the following feasible set:

$$\Omega = \left\{ (\mathbf{f}, \mathbf{e}) \middle| v_a = \sum_{w \in W} \sum_{k \in R_w} f_{wk} \delta_{ak}^w, a \in A, \sum_{k \in R_w} f_{wk} + e_w = \bar{q}_w, e_w, f_{wk} \geq 0 \text{ , } w \in W \right\} \tag{23}$$

can satisfy the following conditions:

$$\begin{aligned} p_{wk}\left(U_w - \bar{\mathbf{c}}_w(\mathbf{f}) - \bar{\mathbf{d}}_w(\mathbf{f}_w), 0 - \bar{d}_w(e_w)\right) = \frac{f_{wk}}{q_w}, k \in R_w \\ p_{ww}\left(U_w - \bar{\mathbf{c}}_w(\mathbf{f}) - \bar{\mathbf{d}}_w(\mathbf{f}_w), 0 - \bar{d}_w(e_w)\right) = \frac{e_w}{q_w}, w \in W \end{aligned} \tag{24}$$

where $p_{wk}\left(U_w - \bar{\mathbf{c}}_w(\mathbf{f}) - \bar{\mathbf{d}}_w(\mathbf{f}_w), 0 - \bar{d}_w(e_w)\right)$ and $p_{ww}\left(U_w - \bar{\mathbf{c}}_w(\mathbf{f}) - \bar{\mathbf{d}}_w(\mathbf{f}_w), 0 - \bar{d}_w(e_w)\right)$ are respectively the possibility that path $k \in R_w$ or the dummy link connecting OD pair $w \in W$ is perceived by users as the one with maximal utility.

We proceed to develop a minimization model for probit-based SUE problem with threshold constraints and elastic demand, which is inspired by Meng et al. (2008)'s model:

$$\begin{aligned} \min z_2(\mathbf{f}, \mathbf{e}) = -\sum_{w \in W} \bar{q}_w S_w\left(\bar{U}_w - \bar{\mathbf{c}}_w(\mathbf{f}) - \bar{\mathbf{d}}_w(\mathbf{f}_w), 0 - \bar{d}_w(e_w)\right) - \sum_{w \in W} \sum_{k \in R_w} \bar{d}_{kw}(\mathbf{f}_w) f_{kw} \\ - \sum_{w \in W} \bar{d}_w(e_w) e_w \end{aligned} \tag{25}$$

$$v_a \leq \bar{H}_a, a \in \bar{A} \tag{26}$$

$$v_a = \sum_{w \in W} \sum_{k \in R_w} f_{wk} \delta_{ak}^w, a \in A \tag{27}$$

$$f_{wk} \geq 0, k \in R_w, w \in W \tag{28}$$

$$e_w \geq 0, w \in W \tag{29}$$

where the row vector $\mathbf{e} = (e_w, w \in W)$. Applying the Karush-Kuhn-Tucker (KKT) conditions on this model, it can be shown that any local minimum solution of the linearly constrained minimization model (25)-(29) - $(\mathbf{f}^*, \mathbf{e}^*) = \left(f_{wk}^*, k \in R_w, w \in W; e_w^*, w \in W\right)$- satisfies the following conditions:

$$f_{wk}^* = \bar{q}_w \times p_{wk}\left(\bar{U}_w - \mathbf{c}_w(\mathbf{f}^*) - \boldsymbol{\lambda}_w^*, 0\right), k \in R_w, w \in W \tag{30}$$

$$e_w^* = \bar{q}_w \times p_{ww}\left(\bar{U}_w - \mathbf{c}_w(\mathbf{f}^*) - \boldsymbol{\lambda}_w^*, 0\right), w \in W \tag{31}$$

$$v_a^* = \sum_{w \in W} \sum_{k \in R_w} f_{wk}^* \delta_{ak}^w, a \in A \tag{32}$$

$$\mu_a^* \times \left(v_a^* - \bar{H}_a\right) = 0, \quad a \in \bar{A} \tag{33}$$

$$\mu_a^* \geq 0, a \in \bar{A} \tag{34}$$

where μ_a^* is the optimal Lagrangian multiplier with respect to the threshold constraint (23); the row path travel vector $\mathbf{c}_w(\mathbf{f}^*) = \left(c_{wk}(\mathbf{f}^*) : k \in R_w\right)$ for OD pair $w \in W$, and the row vector $\boldsymbol{\lambda}_w^* = \left(\lambda_{wk}^* : k \in R_w\right)$ with elements:

$$\lambda_{wk}^* = \sum_{a \in \bar{A}} \mu_a^* \delta_{ak}^w, k \in R_k, w \in W \tag{35}$$

Note that λ_{wk}^*, shown in eqn. (35), is the sum of the optimal Lagrangian multipliers of those entries on the path $k \in R_w$ between OD pair $w \in W$.

Eqns. (30) and (31) implies that the local minimum solution $\left(\mathbf{f}^*, \mathbf{e}^*\right)$ satisfies the probit-based SUE conditions (Sheffi, 1985). Moreover, when all the path flows and the excess-demand e_w are summed up, we have

$$\sum_{k \in R_w} f_{wk}^* + e_w^* = \bar{q}_w \times \left[p_{wk}\left(\bar{U}_w - \mathbf{c}_w\left(\mathbf{f}^*\right) - \boldsymbol{\lambda}_w^*, 0\right) + p_{ww}\left(\bar{U}_w - \mathbf{c}_w\left(\mathbf{f}^*\right) - \boldsymbol{\lambda}_w^*, 0\right) \right] = \bar{q}_w \tag{36}$$

Thus, the flow-demand conservation equation for elastic demand problem can be automatically satisfied. According to the right hand side of eqn. (30), it can be seen that the travel impedance on path k contains an addition proportion λ_{wk}^*, which equals to the summation of optimal Lagrangian multipliers of those entries. Based on these optimal Lagrangian multipliers, we can define an entry-specific toll pattern.

$$\tau_a^* = \mu_a^* \times \beta, a \in \bar{A} \tag{37}$$

Thus, the generalized actual path travel time at the probit-based SUE status can be rewritten by

$$\tilde{c}_{wk}^* = c_{wk}^* + \lambda_{wk}^* = \sum_{a \in A} t_a\left(v_a^*\right)\delta_{ak}^w + \sum_{a \in \bar{A}} \left(\tau_a^* \big/ \beta \right) \delta_{ak}^w, k \in R_w, w \in W \tag{38}$$

In light of eqns. (32)-(34), (37) and (38), it can be concluded that the entry-specific toll pattern $\boldsymbol{\tau}^*$ is the effective toll solution because it satisfies the conditions:

$$v_a^* \le \bar{H}_a, \quad a \in \bar{A} \tag{39}$$

$$\tau_a^* \times \left(v_a^* - \bar{H}_a \right) = 0, \quad a \in \bar{A} \tag{40}$$

$$\tau_a^* \ge 0, a \in \bar{A} \tag{41}$$

In other words, the relation between the effective toll solution and the optimal Lagrangian multipliers does hold for the probit-based SUE problem. Therefore, the effective toll solution can be determined easily according to eqn. (37) if we can obtain the optimal Lagrangian multipliers of minimization model (25)-(29).

3.2 Lagrangian Dual Model and Its Gradient Projection Method

In order to find the optimal Lagrangian multipliers with respect threshold constraint (26) for the minimization model (25)-(29), we can exploit its Lagrangian dual formulation presented below (Bazaraa et al., 1993).

$$\max_{\boldsymbol{\mu} \ge 0} \varphi\left(\boldsymbol{\mu}\right) \tag{42}$$

where the row vector $\boldsymbol{\mu} = \left(\mu_a : a \in \bar{A}\right)$ in which μ_a is the Lagrangian multiplier of threshold constraint (26), and the concave function $\varphi\left(\boldsymbol{\mu}\right)$ is defined as follows.

$$\varphi\left(\boldsymbol{\mu}\right) = \min_{(\mathbf{f}, \mathbf{e}) \in \Omega} \left[z_2\left(\mathbf{f}, \mathbf{e}\right) + \sum_{a \in \bar{A}} \mu_a \left(v_a - \bar{H}_a\right) \right] \tag{43}$$

Given a Lagrangian multiplier vector $\boldsymbol{\mu}$, the right hand side of eqn. (43) is a minimization model which amounts to the following minimization model in terms of the optimal solution, because the term $\sum_{a\in\bar{A}}\mu_a\bar{H}_a$ shown in eqn. (43) is a constant.

$$\min_{(\mathbf{f},\mathbf{e})\in\Omega}\left[z_2(\mathbf{f},\mathbf{e})+\sum_{a\in\bar{A}}\mu_a v_a\right] \tag{44}$$

KKT conditions of minimization model (44) shows that any local minimum solution satisfies the probit-based SUE conditions with elastic demand, in terms of the following generalized link travel time functions:

$$\hat{t}_a(v_a)=\begin{cases}t_a(v_a)+\mu_a, a\in\bar{A}\\ t_a(v_a), a\in A\setminus\bar{A}\end{cases},\quad a\in A \tag{45}$$

For any given $\boldsymbol{\mu}$, this link travel cost function is evidently strictly monotone and continuously differentiable, which is a sufficient condition for the uniqueness of the resultant optimal link flow solution (Canteralla, 1997), denoted by $\{v_a(\boldsymbol{\mu}), a\in A\}$. According to Theorem 6.3.3 of Bazaraa et al. (1993), the uniqueness of the optimal link flow solution implies that $\varphi(\boldsymbol{\mu})$ is a continuously differentiable function with the gradient:

$$\nabla\varphi(\boldsymbol{\mu})=\left(v_a(\boldsymbol{\mu})-\bar{H}_a : a\in\bar{A}\right) \tag{46}$$

Hence, the Lagrangian dual formulation (42) is a concave continuously differentiable maximization model, which can be efficiently solved by a global convergent gradient projection method with the iterative solution updating scheme:

$$\boldsymbol{\mu}^{(n+1)}=P_{\Re_+^{|\bar{A}|}}\left[\boldsymbol{\mu}^{(n)}+\alpha_n\varphi\left(\boldsymbol{\mu}^{(n)}\right)\right] \tag{47}$$

where n is the number of iterations; $|\bar{A}|$ is the number of entries; $\Re_+^{|\bar{A}|}$ is the $|\bar{A}|$-dimensional nonnegative space; and the projection operation $P_{\Re_+}[\cdot]$ is defined by

$$P_{\Re_+}[\mathbf{y}]=\arg\min_{\mathbf{x}\in\Re^+}\sum_{a\in\bar{A}}(x_a-y_a)^2 \tag{48}$$

In eqn. (47), $\{\alpha_n\}$ is a predetermined step size sequence satisfying the three conditions:

$$0<\alpha_n<1 \text{ and } \lim_{n\to\infty}\alpha_n=0 \tag{49}$$

$$\sum_{n=1}^{\infty}\alpha_n=+\infty \tag{50}$$

$$\sum_{n=1}^{\infty}\alpha_n^2<\infty \tag{51}$$

There are a few step size sequences fulfilling the above conditions; for example

$$\alpha_n=\frac{\rho}{n}, n=1,2,\cdots,\infty \tag{52}$$

where parameter $0<\rho<1$.

The Lagrangian multiplier updating formula shown in eqn. (47) can be rewritten as follows:

$$\mu_a^{(n+1)}=\max\left\{0,\mu_a^{(n)}+\alpha_n\left(v_a\left(\boldsymbol{\mu}^{(n)}\right)-\bar{H}_a\right)\right\}, a\in\bar{A} \tag{53}$$

Larsson et al. (1996) have already shown that the gradient project method is globally convergent to the optimal Lagrangian multipliers, namely,

$$\boldsymbol{\mu}^{(n)}\to\boldsymbol{\mu}^* \tag{54}$$

More interestingly, according to Proposition 5.1 of Bertsekas and Tsitsiklis (1989), $\left\{\mu_a^{(n+1)} : a \in \bar{A}\right\}$ is the optimal Lagrangian multiplier solution, if it satisfies the following condition

$$\boldsymbol{\mu}^{(n)} = \boldsymbol{\mu}^{(n+1)} = P_{\Re_+^{|\bar{A}|}}\left[\boldsymbol{\mu}^{(n)} + \alpha_n \varphi\left(\boldsymbol{\mu}^{(n)}\right)\right] \tag{55}$$

4. THE TRIAL-AND-ERROR METHOD BASED ON THE ENTRY-SPECIFIC TRAFFIC COUNTS

Link travel time functions, O-D demand functions and VOT, which are involved in the mathematical models built for the effective toll solution, are not all readily available in practice. Without accurate information for these functions and the VOT, we can not employ the existing optimization algorithms to find the effective toll solution. However, given an entry-specific toll pattern, traffic counts on each entry in the cordon-based congestion pricing scheme can be easily obtained. These entry-specific traffic counts are the driver's response result to the entry-specific toll pattern. By comparing traffic counts on an entry with its predetermined threshold, we can adjust the entry-specific toll pattern in order to obtain the effective toll solution of the cordon-based congestion pricing. Based on the entry-specific traffic counts, a trial-and-error method can be elaborated as follows:

Step 0 (Initialization) Set a rational entry-specific toll pattern, denoted by $\boldsymbol{\tau}^{(1)} = \left(\tau_a^{(1)} : a \in \bar{A}\right)$, on the entries. Take a predetermined step size sequence $\left\{\alpha_n, n = 1, 2, \cdots\right\}$ satisfying conditions (49)-(51). Let the number of iterations $n = 1$.

Step 1 (Obtain traffic counts on each entry) After implementation of the entry-specific toll pattern $\boldsymbol{\tau}^{(n)}$, count the number of vehicles passing though each entry during one peak hour, denoted by $v_a^{(n)}, a \in \bar{A}$.

Step 2 (Update the entry-specific toll pattern) Adjust the toll imposed on each entry according to the formula:

$$\tau_a^{(n+1)} = \max\left\{0, \tau_a^{(n)} + \alpha_n\left(v_a^{(n)} - \bar{H}_a\right)\right\}, a \in \bar{A} \tag{56}$$

Step 3 (Check the stop criterion) If the following stopping criterion shown in eqn. (57) is met, stop and output the effective toll solution $\boldsymbol{\tau}^{(n+1)} = \left(\tau_a^{(n+1)} : a \in \bar{A}\right)$. Otherwise, set $n = n+1$ and go to Step 1.

$$\max_{a \in \bar{A}}\left\{\left|\tau_a^{(n+1)} - \tau_a^{(n)}\right|\right\} \le \varepsilon \tag{57}$$

where ε is the predetermined tolerance.

The above trial-and-error method only needs traffic counts at each entry. Since we assume that behaviour of drivers in route choice obeys the probit-based SUE principle, traffic counts on each entry $a \in \bar{A}$ obtained in Step 1, $\left\{v_a^{(n)}, a \in \bar{A}\right\}$, is the probit-based SUE link flow with respect to the following generalized actual link travel time functions:

$$\tilde{t}_a(v_a) = \begin{cases} t_a(v_a) + \left(\tau_a^{(n)}/\beta\right), a \in \bar{A} \\ t_a(v_a), a \in A \setminus \bar{A} \end{cases}, \quad a \in A \tag{58}$$

Comparing eqn. (45) with (58), it can be concluded that $\left\{v_a^{(n)}, a \in \bar{A}\right\}$ is the optimal link flow resulting from the optimal solution of minimization model (44) with the Lagrangian multipliers: $\mu_a = \tau_a^{(n)}/\beta, a \in \bar{A}$. In other words, we have

$$v_a^{(n)} = v_a\left(\boldsymbol{\tau}^{(n)}/\beta\right), a \in \bar{A} \tag{59}$$

where $v_a\left(\boldsymbol{\tau}^{(n)}/\beta\right), a \in \bar{A}$ is the optimal link traffic flows of the minimization model (44) in terms of $\mu_a = \tau_a^{(n)}/\beta, a \in \bar{A}$. According to eqn. (46), the toll updating formula shown in eqn. (56) can be rewritten below.

$$\frac{\tau_a^{(n+1)}}{\beta} = \max\left\{0, \frac{\tau_a^{(n)}}{\beta} + \frac{\alpha_n}{\beta}\left(v_a\left(\boldsymbol{\tau}^{(n)}/\beta\right) - \bar{H}_a\right)\right\}, a \in \bar{A} \tag{60}$$

Despite the existence of VOT β in eqn. (60), after a number of trials we have $\frac{\alpha_n}{\beta} < 1$ even if $\beta < 1.0$. Referring to eqn. (53), it can be seen that the iterative updating scheme shown in (60) is the gradient projection method for solving the Lagrangian dual model (42). Thus, we have

$$\boldsymbol{\tau}^{(n)} \to \beta \times \boldsymbol{\mu}^* \tag{61}$$

In other words, the trial-and-error method is able to find the effective toll solution with the probit-based SUE constraint. Moreover, the stopping criterion expressed by eqn. (57) is justifiable based on the conclusion presented above in eqn. (55).

5. A NUMERICAL SIMULATION EXAMPLE

To test the proposed trial-and-error method for estimating effective toll solution with the probit-based SUE constraints, we use the Singapore electronic road pricing system (ERP) on the downtown Orchard Road, shown in Figure 1, as a numerical example. Figure 1 is downloaded from the website of Singapore Land Transport Authority and it clearly shows that the ERP on Orchard Road is a typical cordon-based congestion pricing scheme. The Orchard cordon comprises 12 entries with their names shown at the lower left corner of Figure 1. The Orchard Road network in Singapore can be topologized as a graph shown in Figure 2, in which there are 33 nodes, 104 links and 12 OD pairs. The 12 entries are represented by those links with bold line segments in Figure 2, namely,

$$\bar{A} = \begin{Bmatrix} 5 \to 13, 6 \to 14, 7 \to 15, 8 \to 17, 18 \to 17, 26 \to 25, 32 \to 24, 31 \to 23, \\ 30 \to 22, 29 \to 20, 28 \to 19, 10 \to 11 \end{Bmatrix} \tag{62}$$

The tolls imposed on these 12 entries aims to mitigate traffic congestion in the Orchard Road area represented by links shown in the rectangle determined by nodes 11, 17, 19 and 25.

FIGURE 1 ERP system on the Orchard Road of Singapore

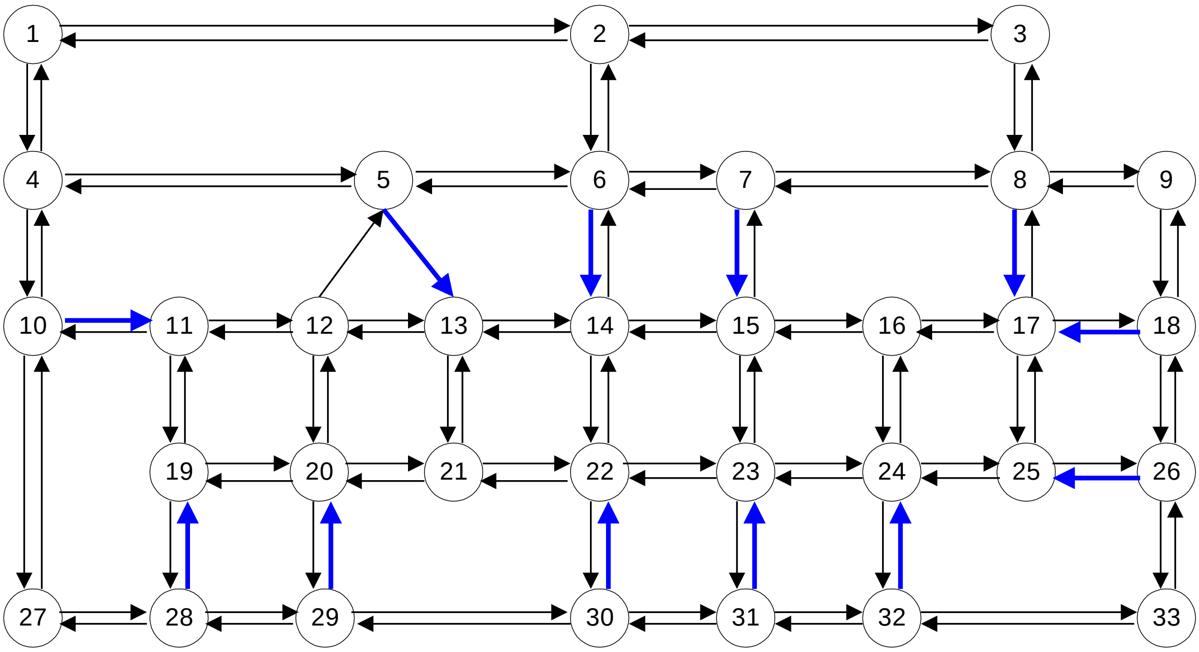

FIGURE 2 Topological structure of the Orchard Road network

5.1 Simulation of Entry-specific Traffic Counts on Each Entry

For the Orchard Road example with network structure shown in Figure 2, the key component of implementing the proposed trial-and-error method is how to get the traffic counts on each of these 12 entries. These traffic counts can be automatically obtained from the electronic device installed on the ERP gantry at each entry. Note that for the numerical example here, the traffic counts on each entry can be simulated by performing a probit-based SUE traffic assignment with the generalized travel time functions expressed by eqn. (58). The method of successive average (MSA) is employed for solving this probit-based SUE traffic assignment problem. MSA only requires a probit-based stochastic network loading procedure at each iteration. We use the Monte Carlo simulation based method (Sheffi, 1985) to implement this probit-based stochastic network loading procedure. Stopping criterion for MSA is the error between the successive averages of the last 3 iterations; see equations (12.52) and (12.53) of Sheffi (1985).

To simulate traffic counts on each entry after a trial of entry-specific toll pattern, it is assumed that travel time on each link follows the BPR function:

$$t_a\left(v_a\right)=t_a^0\left(1.0+0.15\left(\frac{v_a}{H_a}\right)^4\right),a\in A \tag{63}$$

where the free-flow travel time, t_a^0 and physical capacity of link flows H_a are tabulated in Table 1. It should be noted that threshold constraint of any link flow can not exceed its physical capacity, i.e. $\bar{H}_a \le H_a, a\in \bar{A}$ Regarding the OD demand, initial travelling utility $\bar{U}_w$ and total travel demand $\bar{q}_w$ of all the 12 OD pairs are enclosed in Table 2.

It is further assumed that the VOT $\beta=10$ Singapore-Dollars/hour and the parameter in the covariance, shown in eqn.(6), $\theta=1.0$.

TABLE 1 Link Data

Link No.	**Start node**	**End node**	**Free-flow travel time** t_a^0 **(Seconds)**	**Number of lanes**	**Capacity** H_a **(vehicles/hours)**
1	2	1	60	3	5400
2	1	2	60	3	5400
3	3	2	100	3	5400
4	2	3	100	3	5400
5	1	4	40	3	5400
6	4	1	40	2	5400
7	5	4	90	2	3600
8	4	5	90	2	3600
9	6	5	42	2	3600
10	5	6	42	2	3600
11	2	6	80	4	7200
12	6	2	80	4	7200
13	3	8	72	2	3600
14	8	3	72	2	3600
15	7	6	160	1	1800
16	6	7	160	1	1800
17	8	7	120	1	1800
18	7	8	120	1	1800
19	9	8	60	2	3600
20	8	9	60	2	3600
21	4	10	80	3	5400
22	10	4	80	3	5400
23	12	5	55	2	3600
24	5	13	55	2	3600
25	6	14	80	4	7200
26	14	6	80	4	7200

27	7	15	120	1	1800
28	15	7	120	1	1800
29	8	17	60	2	3600
30	17	8	60	2	3600
31	9	18	90	2	3600
32	18	9	90	2	3600
33	11	10	16	3	5400
34	10	11	16	3	5400
35	12	11	40	3	5400
36	11	12	40	3	5400
37	13	12	24	3	5400
38	12	13	24	3	5400
39	14	13	48	3	5400
40	13	14	48	3	5400
41	14	15	40	3	5400
42	15	14	40	3	5400
43	16	15	12	3	5400
44	15	16	12	3	5400
45	17	16	60	3	5400
46	16	17	60	3	5400
47	18	17	60	3	5400
48	17	18	60	3	5400
49	10	27	20	3	5400
50	27	10	20	3	5400
51	11	19	24	1	1800
52	19	11	24	1	1800
53	12	20	20	3	5400
54	20	12	20	3	5400
55	13	21	30	2	3600
56	21	13	30	2	3600
57	14	22	12	4	7200
58	22	14	12	4	7200
59	23	15	12	2	3600
60	15	23	12	2	3600
61	16	24	12	2	3600
62	24	16	12	2	3600
63	17	25	14	2	3600
64	25	17	14	2	3600
65	18	26	20	3	5400
66	26	18	20	3	5400
67	20	19	80	1	1800
68	19	20	80	1	1800
69	20	21	20	3	5400
70	21	20	20	3	5400
71	22	21	48	3	5400
72	21	22	48	3	5400
73	23	22	20	3	5400
74	22	23	20	3	5400
75	24	23	28	3	5400
76	23	24	28	3	5400
77	25	24	60	3	5400
78	24	25	60	3	5400
79	26	25	60	3	5400

80	25	26	60	3	5400
81	19	28	24	1	1800
82	28	19	24	1	1800
83	20	29	12	3	5400
84	29	20	12	3	5400
85	22	30	18	4	7200
86	30	22	18	4	7200
87	23	31	16	2	3600
88	31	23	16	2	3600
89	24	32	18	3	5400
90	32	24	18	3	5400
91	26	33	60	3	5400
92	33	26	60	3	5400
93	28	27	20	3	5400
94	27	28	20	3	5400
95	29	28	40	3	5400
96	28	29	40	3	5400
97	30	29	90	2	3600
98	29	30	90	2	3600
99	31	30	30	2	3600
100	30	31	30	2	3600
101	32	31	60	2	3600
102	31	32	60	2	3600
103	33	32	96	3	5400
104	32	33	96	3	5400

TABLE 2 Parameters involved in the travel demand function for each OD pair

OD pair w	**Initial Traveling Utility** $\bar{U}_w$ **(cents)**	**Total travel demand** $\bar{q}_w$ **(vehicles/hours)**
$1 \rightarrow 33$	145	5000
$9 \rightarrow 1$	163	4000
$3 \rightarrow 27$	132	5000
$27 \rightarrow 9$	146	5000
$2 \rightarrow 29$	113	6000
$18 \rightarrow 28$	133	6000
$4 \rightarrow 24$	108	3000
$32 \rightarrow 14$	50	5000
$33 \rightarrow 3$	143	5000
$25 \rightarrow 4$	145	5000
$28 \rightarrow 6$	65	3000
$7 \rightarrow 23$	39	2000

5.2 Numerical Results and Discussions

To assess the impact of the threshold constraints on the effective toll solution, two scenarios of the threshold values, shown in Table 3, are examined. The initial toll pattern is set to be zero on all the 12 entries and stopping tolerance is $\varepsilon = 0.01$. Subsequently, trial-and-error method is adopted to calculate the effective toll solutions of these two scenarios in turn. Table 4 gives the effective toll solution obtained by trial-and-error method for the two scenarios. According to Table 4, it can be seen that the estimated effective toll solution indeed satisfies the three conditions shown in eqn. (8)-(10), i.e. the non-negative toll charge τ_a^* would take a positive value only if the flow-capacity ratio $v_a/\bar{H}_a$ is close to 1.0. Interestingly, the toll charge is zero for the entries 27, 86 and 88 because their link flows are strictly lower than their thresholds. In addition, the effective toll solution for Scenario 1 is greater than for Scenario 2, because the threshold values for the Scenario 1 are smaller than that for Scenario 2, namely Scenario 1 is more rigorous than Scenario 2.

TABLE 3 Two Scenarios for the thresh hold constraints

Entry ($a \in \bar{A}$)	Scenario 1 Threshold ($\bar{H}_a$) (vehicles/hour)	Scenario 2 Threshold ($\bar{H}_a$) (vehicles/hour)
24	2600.0	3600.0
25	2600.0	4000.0
27	1800.0	1800.0
29	2600.0	2500.0
34	2600.0	2500.0
47	2600.0	3700.0
79	2600.0	3700.0
82	1800.0	1800.0
84	2600.0	2500.0
86	2600.0	2000.0
88	2600.0	2000.0
90	2600.0	3500.0

TABLE 4 The effective toll solutions

Entry No.	Scenario 1		Scenario 2	
($a \in \bar{A}$)	$v_a/\bar{H}_a$	τ_a^*	$v_a/\bar{H}_a$	τ_a^*
24	0.99	9.46	1.00	2.92
25	0.99	8.15	0.99	2.70
27	0.78	0.00	0.68	0.00
29	0.99	4.12	1.00	3.20
34	1.02	3.45	1.02	2.07
47	0.99	9.22	1.01	3.90
79	0.99	7.81	1.01	2.95
82	0.99	1.69	0.98	1.12
84	1.00	2.74	1.02	2.53
86	0.85	0.00	0.90	0.00
88	0.81	0.00	0.82	0.00
90	1.01	5.90	1.00	2.30

To examine the impact of the predetermined step size sequence on the convergent speed of the trial-and-error method, three step size sequences - $\{0.01/n\}$, $\{0.015/n\}$ and $\{0.03/n\}$ - are investigated. Without loss of generality, Scenario 1 of threshold constraints is chosen for this test. Figure 3 depicts the convergent trend of the trial-and-error method using these three step size sequences. According to Figure 3, it can be seen that (a) these three sets of tests can all converge; (b) the convergent speed is indeed influenced by predetermined step size sequence, and smaller step size sequence would contribute to a faster convergent speed.

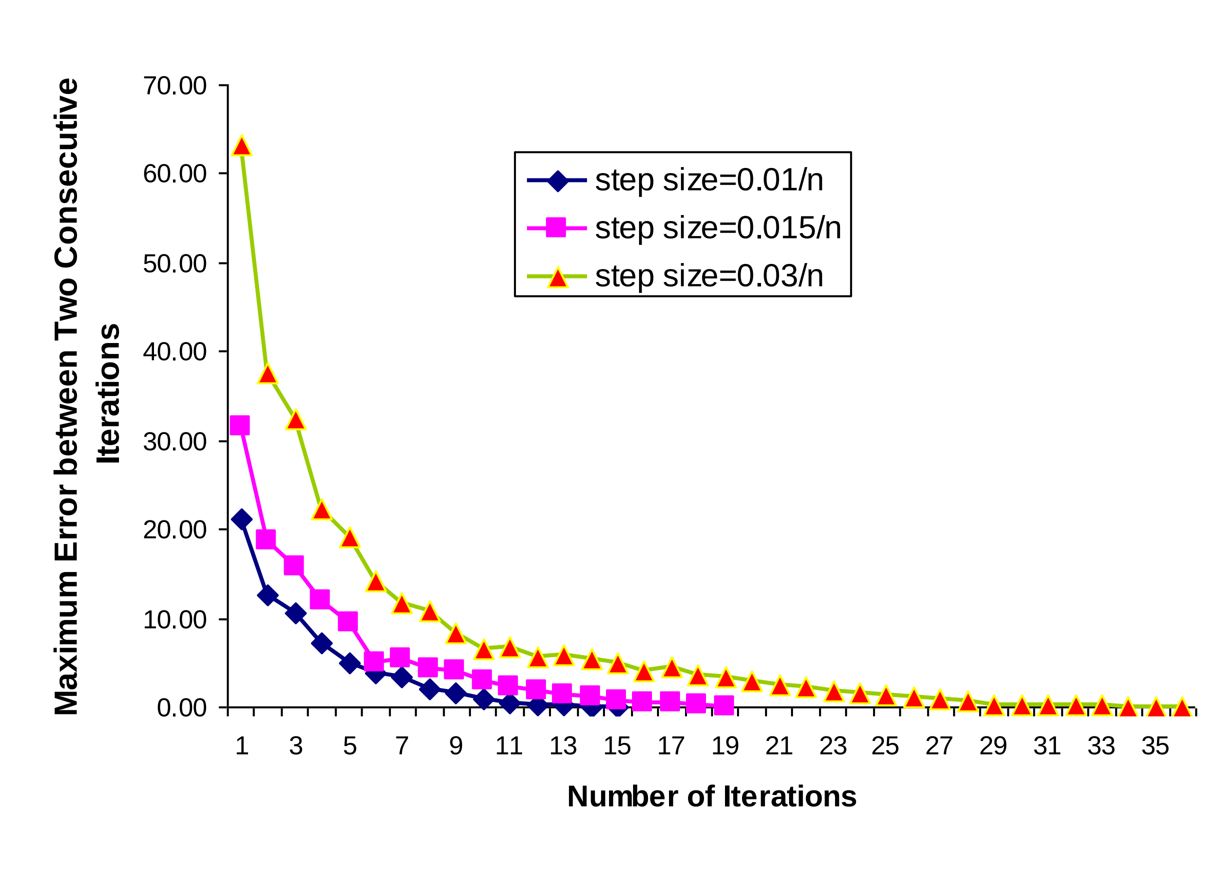

FIGURE 3 Convergent trend of the trial-and-error method with three step size sequences

6. CONCLUSIONS

This paper has proposed a convergent trial-and-error method only based on the entry-specific traffic counts, to estimate the effective toll solution of cordon-based congestion pricing problem with probit-based SUE constraints. This trial-and-error method is a variation of the gradient project method, which can solve a proposed model for probit-based SUE problem with the threshold constraints and elastic demand. This is because the effective toll solution divided by the VOT is equal to the optimal Lagrangian multipliers with respect to the threshold constraints. Since this proposed trial-and-error method does not rely on the link travel time functions, OD demand functions or VOT, it is suitable for the adjustment of entry-specific toll pattern for congestion pricing operators in practice. This paper also enriches the congestion pricing theory by demonstrating the availability that trial-and-error method does work for the cordon-based congestion pricing scheme when the behaviour of drivers follows probit-based SUE principle.

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